The Finite Element Method for the Analysis of Non-Linear and Dynamic Systems

Prof. Dr. Michael Havbro Faber

Swiss Federal Institute of Technology
ETH Zurich, Switzerland
Contents of Today's Lecture

• Motivation, overview and organization of the course

• Introduction to non-linear analysis

• Formulation of the continuum mechanics incremental equations of motion
Motivation, overview and organization of the course

• Motivation

In FEM 1 we learned about the **steady state analysis** of linear systems

however,

the systems we are dealing with in structural engineering are generally not steady state and also not linear

We must be able to assess the need for a particular type of analysis and we must be able to perform it
Motivation, overview and organization of the course

• Motivation

What kind of problems are not steady state and linear?

E.g. when the:

material behaves non-linearly

deflections become big (p-Δ effects)

loads vary fast compared to the eigenfrequencies of the structure

General feature: Response becomes load path dependent
Motivation, overview and organization of the course

• Motivation

What is the “added value” of being able to assess the non-linear non-steady state response of structures?

E.g. assessing the:

- structural response of structures to extreme events (rock-fall, earthquake, hurricanes)

- performance (failures and deformations) of soils

- verifying simple models
Motivation, overview and organization of the course

• Collapse Analysis of the World Trade Center
Motivation, overview and organization of the course

- Collapse Analysis of the World Trade Center
Motivation, overview and organization of the course

- Analysis of ultimate collapse capacity of jacket structure
Motivation, overview and organization of the course

- Analysis of ultimate collapse capacity of jacket structure
Motivation, overview and organization of the course

• Analysis of soil performance
Motivation, overview and organization of the course

- Analysis of bridge response

Mode 1: 0.356257 Hz
Mode 2: 0.422196 Hz
Mode 3: 0.429367 Hz
Mode 4: 0.468655 Hz
Motivation, overview and organization of the course

Steady state problems (Linear/Non-linear):

The response of the system does not change over time

\[ KU = R \]

Propagation problems (Linear/Non-linear):

The response of the system changes over time

\[ M\ddot{U}(t) + C\dot{U}(t) + KU(t) = R(t) \]

Eigenvalue problems:

No unique solution to the response of the system

\[ Av = \lambda Bv \]
Motivation, overview and organization of the course

• Organization

The lectures will be given by:

M. H. Faber

Exercises will be organized/attended by:

Jianjun Qin

By appointment, HIL E13.1.
Motivation, overview and organization of the course

• Organization

PowerPoint files with the presentations will be uploaded on our homepage one day in advance of the lectures

http://www.ibk.ethz.ch/fa/education/FE_II

The lecture as such will follow the book:

"Finite Element Procedures" by K.J. Bathe, Prentice Hall, 1996
Motivation, overview and organization of the course

• Overview

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<thead>
<tr>
<th>Date</th>
<th>Pages</th>
<th>Subject</th>
</tr>
</thead>
<tbody>
<tr>
<td>18.09.2009</td>
<td>485-502</td>
<td>Non-linear Finite Element Calculations in solids and structural mechanics</td>
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<td>- Introduction to non-linear calculations</td>
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<td>- The incremental approach to continuum mechanics</td>
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<td>25.09.2009</td>
<td>502-528</td>
<td>Non-linear Finite Element Calculations in solids and structural mechanics</td>
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<td>- Deformation gradients, strain and stress tensors</td>
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<td>- The Langrangian formulation – only material non-linearity</td>
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<td>02.10.2009</td>
<td>538-548</td>
<td>Non-linear Finite Element Calculations in solids and structural mechanics</td>
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<td>- Displacement based iso-parametric finite elements in continuum mechanics</td>
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<td>09.10.2009</td>
<td>548-560</td>
<td>Non-linear Finite Element Calculations in solids and structural mechanics</td>
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<td>- Displacement based iso-parametric finite elements in continuum mechanics</td>
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Motivation, overview and organization of the course

• Overview

16.10.2009  561-578  Non-linear Finite Element Calculations in solids and structural mechanics
- Total Langrangian formulation
- Extended Lagrangian formulation
- Structural elements

23.10.2009  581-617  Non-linear Finite Element Calculations in solids and structural mechanics
- Introduction to constitutive relations
- Non-linear constitutive relations

30.10.2009  622-640  Non-linear Finite Element Calculations in solids and structural mechanics
- Contact problems
- Practical considerations

06.11.2009  768-784  Dynamical Finite Element Calculations
- Introduction
- Direct integration methods
Motivation, overview and organization of the course

Overview

13.11.2009  785-800  Dynamical Finite Element Calculations
- Mode superposition

20.11.2009  801-815  Dynamical Finite Element Calculations
- Analysis of direct integration methods

27.11.2009  824-830  Dynamical Finite Element Calculations
- Solution of dynamical non-linear problems

04.12.2009  887-910  Solution of Eigen value problems
- The vector iteration method

11.12.2009  911-937  Solution of Eigen value problems
- The transformation method

18.12.2009  Introduction to FEM-software
Introduction to non-linear analysis

• Previously we considered the solution of the following linear and static problem:

\[ KU = R \]

for these problems we have the convenient property of linearity, i.e:

\[ KU = \lambda R, \quad \lambda = 1 \]

↓

\[ U^* = \lambda U, \quad \lambda \neq 1 \]

If this is not the case we are dealing with a non-linear problem!
Introduction to non-linear analysis

• Previously we considered the solution of the following linear and static problem:

\[ KU = R \]

we assumed:

small displacements when developing the stiffness matrix \( K \) and the load vector \( R \), because we performed all integrations over the original element volume

that the \( B \) matrix is constant independent of element displacements

the stress-strain matrix \( C \) is constant

boundary constraints are constant
# Introduction to non-linear analysis

## Classification of non-linear analyses

<table>
<thead>
<tr>
<th>Type of analysis</th>
<th>Description</th>
<th>Typical formulation used</th>
<th>Stress and strain measures used</th>
</tr>
</thead>
<tbody>
<tr>
<td>Materially-nonlinear only</td>
<td>Infinitesimal displacements and strains; stress train relation is non-linear</td>
<td>Materially-nonlinear-only (MNO)</td>
<td>Engineering strain and stress</td>
</tr>
<tr>
<td>Large displacements, large rotations but small strains</td>
<td>Displacements and rotations of fibers are large; but fiber extensions and angle changes between fibers are small; stress strain relationship may be linear or non-linear</td>
<td>Total Lagrange (TL)</td>
<td>Second Piola-Kirchoff stress, Green-Lagrange strain</td>
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<tr>
<td></td>
<td></td>
<td>Updated Lagrange (UL)</td>
<td>Cauchy stress, Almansi strain</td>
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<td></td>
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<td>Updated Lagrange (UL)</td>
<td>Cauchy stress, Logarithmic strain</td>
</tr>
</tbody>
</table>
Introduction to non-linear analysis

- Classification of non-linear analyses

\[ \Delta \]

\[ P \]

\[ \frac{P}{2} \]

\[ L \]

\[ \frac{P}{2} \]

\[ L \]

\[ \sigma = \frac{P}{A} \]

\[ \varepsilon = \frac{\sigma}{E} \]

\[ \Delta = \varepsilon L \]

Linear elastic (infinitesimal displacements)
Introduction to non-linear analysis

- Classification of non-linear analyses

\[ \varepsilon = \frac{\sigma_y}{E} + \frac{\sigma - \sigma_y}{E_T} \]

\[ \varepsilon < 0.04 \]

Materially nonlinear only (infinitesimal displacements, but nonlinear stress-strain relation)
Introduction to non-linear analysis

- Classification of non-linear analyses

Large displacements and large rotations but small strains (linear or nonlinear material behavior)
Introduction to non-linear analysis

- Classification of non-linear analyses

Large displacements, large rotations and large strains (linear or nonlinear material behavior)
Introduction to non-linear analysis

- Classification of non-linear analyses

Chang in boundary conditions
Introduction to non-linear analysis

- Example: Simple bar structure

\[ \begin{align*}
\text{Area} &= 1 \text{cm}^2 \\
L_a &= 10 \text{cm} \\
L_b &= 5 \text{cm}
\end{align*} \]

\[ \begin{align*}
\sigma &= \text{yield stress} \\
\epsilon &= \text{yield strain}
\end{align*} \]

\[ \begin{align*}
E &= 10^7 \text{ N/cm}^2 \\
E_T &= 10^5 \text{ N/cm}^2 \\
\sigma_y : &\text{ yield stress} \\
\epsilon_y : &\text{ yield strain}
\end{align*} \]
Introduction to non-linear analysis

- Example: Simple bar structure

\[ t \varepsilon_a = \frac{t u}{L_a}, \quad t \varepsilon_b = -\frac{t u}{L_b} \]

\[ t R + t \sigma_b A = t \sigma_a A \]

\[ t \varepsilon = \frac{t \sigma}{E} \quad \text{(elastic region)} \]

\[ t \varepsilon = \varepsilon_y + \frac{t \sigma - \sigma_y}{E} \quad \text{(plastic region)} \]

\[ \Delta \varepsilon = \frac{\Delta \sigma}{E} \quad \text{(unloading)} \]
Introduction to non-linear analysis

- Example: Simple bar structure

Both sections elastic

\[ \sigma_a = \frac{\tau R}{3A}, \sigma_b = -\frac{2\tau R}{3A} \]

\[ \tau R = E A \tau u \left( \frac{1}{L_a} + \frac{1}{L_b} \right) \Rightarrow \tau u = \frac{\tau R}{3 \cdot 10^6} \]
Introduction to non-linear analysis

• Example: Simple bar structure

<table>
<thead>
<tr>
<th>Section a</th>
<th>Section b</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_a = 10 \text{cm}$</td>
<td>$L_b = 5 \text{cm}$</td>
</tr>
</tbody>
</table>

Area = 1 cm$^2$

$E = 10^7 \text{N/cm}^2$

$E_T = 10^5 \text{N/cm}$

$\sigma_y$: yield stress

$\varepsilon$: yield strain

Section a is elastic while section b is plastic

\[ t^* R = \frac{3}{2} \sigma_y A \]

\[ \sigma_a = E \frac{t^* u}{L_a}, \sigma_b = -E_T \left( \frac{t^* u}{L_b} - \varepsilon_y \right) - \sigma_y \]

\[ t^* R = \frac{EA t^* u}{L_a} + \frac{E_T A t^* u}{L_b} - E_T \varepsilon_y A + \sigma_y A \Rightarrow \]

\[ t^* u = \frac{t^* R / A + E_T \varepsilon_y - \sigma_y}{E / L_a + E / L_b} = \frac{t^* R}{1.02 \cdot 10^6} - 1.9412 \cdot 10^{-2} \]
Introduction to non-linear analysis

• What did we learn from the example?

The basic problem in general nonlinear analysis is to find a state of equilibrium between externally applied loads and element nodal forces

\[ tR - tF = 0 \]

\[ tR = tR_B + tR_S + tR_C \]

\[ tF = tR_I \]

\[ tF = \sum_m \int_{V^{(m)}} tB^{(m)T} t\tau^{(m)} t dV^{(m)} \]

We must achieve equilibrium for all time steps when incrementing the loading

Very general approach

includes implicitly also dynamic analysis!
Introduction to non-linear analysis

• The basic approach in incremental analysis is

\[ t + \Delta t \mathbf{R} - (t + \Delta t) \mathbf{F} = 0 \]

assuming that \( t + \Delta t \mathbf{R} \) is independent of the deformations we have

\[ t + \Delta t \mathbf{F} = t \mathbf{F} + \mathbf{F} \]

We know the solution \( t \mathbf{F} \) at time \( t \) and \( \mathbf{F} \) is the increment in the nodal point forces corresponding to an increment in the displacements and stresses from time \( t \) to time \( t + \Delta t \) this we can approximate by

\[ \mathbf{F} = t \mathbf{KU} \]

\[ \mathbf{Tangent \ stiffness \ matrix} \quad t \mathbf{K} = \frac{\partial t \mathbf{F}}{\partial t \mathbf{U}} \]
Introduction to non-linear analysis

• The basic approach in incremental analysis is

We may now substitute the tangent stiffness matrix into the equilibrium relation

\[ \dot{t}KU = \dot{t + \Delta t}R - \dot{t}F \]

\[ \downarrow \]

\[ \dot{t + \Delta t}U = \dot{t}U + U \]

which gives us a scheme for the calculation of the displacements

the exact displacements at time \( t + \Delta t \) correspond to the applied loads at \( t + \Delta t \) however we only determined these approximately as we used a tangent stiffness matrix – thus we may have to iterate to find the solution
Introduction to non-linear analysis

- The basic approach in incremental analysis is

We may use the **Newton-Raphson** iteration scheme to find the equilibrium within each load increment

\[
K^{(i-1)}(i) \Delta U^{(i)} = t^{+\Delta t} R - t^{+\Delta t} F^{(i-1)} \quad \text{(out of balance load vector)}
\]

\[
t^{+\Delta t} U^{(i)} = t^{+\Delta t} U^{(i-1)} + \Delta U^{(i)}
\]

with initial conditions

\[
t^{+\Delta t} U^{(0)} = t^{0} U; \quad t^{+\Delta t} K^{(0)} = t^{0} K; \quad t^{+\Delta t} F^{(0)} = t^{0} F
\]
Introduction to non-linear analysis

- The basic approach in incremental analysis is

It may be expensive to calculate the tangent stiffness matrix and,

in the **Modified Newton-Raphson** iteration scheme it is thus only calculated in the beginning of each new load step

in the **quasi-Newton** iteration schemes the secant stiffness matrix is used instead of the tangent matrix

Method of Finite Elements II
Introduction to non-linear analysis

• We look at the example again – simple bar (two load steps)

\[(tK_a + tK_b)\Delta u^{(i)} = t^{+\Delta t} R - (t^{+\Delta t} F_a^{(i-1)} - t^{+\Delta t} F_b^{(i-1)})\]

\[t^{+\Delta t} u^{(i)} = t^{+\Delta t} u^{(i-1)} + \Delta u^{(i)}\]

with initial conditions

\[t^{+\Delta t} u^{(0)} = t u; \quad t^{+\Delta t} F_a^{(0)} = t F_a \quad t^{+\Delta t} F_b^{(0)} = t F_b\]

\[tK_a = \frac{tCA}{L_a}; \quad tK_b = \frac{tCA}{L_b}\]

\[tC = \begin{cases} & E \quad \text{if section is elastic} \\ & E_T \quad \text{if section is plastic} \end{cases}\]
Introduction to non-linear analysis

• We look at the example again – simple bar

Load step 1: $t = 1$:

$\left( ^0 K_a + ^0 K_b \right) \Delta u^{(1)} = 1 R - \frac{1}{1} F_{a}^{(0)} - \frac{1}{1} F_{b}^{(0)}$

\[ \downarrow \]

$\Delta u^{(1)} = \frac{2 \times 10^4}{10^7 \left( \frac{1}{10} + \frac{1}{5} \right)} = 6.6667 \times 10^{-3}$

Iteration 1: ($i = 1$)

$1 u^{(1)} = 1 u^{(0)} + \Delta u^{(1)} = 6.6667 \times 10^{-3}$

$1 \epsilon_a^{(1)} = \frac{1 u^{(1)}}{L_a} = 6.6667 \times 10^{-4} < \epsilon_Y$ (elastic section!)

$1 \epsilon_b^{(1)} = \frac{1 u^{(1)}}{L_b} = 1.3333 \times 10^{-3} < \epsilon_Y$ (elastic section!)

$1 F_a^{(1)} = 6.6667 \times 10^3$; $1 F_b^{(1)} = 1.3333 \times 10^4$

Convergence in one iteration!

$\left( ^0 K_a + ^0 K_b \right) \Delta u^{(2)} = 1 R - \frac{1}{1} F_{a}^{(1)} - \frac{1}{1} F_{b}^{(1)} = 0$

$1 u = 6.6667 \times 10^{-3}$
Introduction to non-linear analysis

- We look at the example again – simple bar

Load step 2: \( t = 2 \):

\[
(1K_a + 1K_b)\Delta u^{(1)} = 2R - 2F_a^{(0)} - 2F_b^{(0)}
\]

\[
\Delta u^{(1)} = \frac{(4 \times 10^3) - (6.6667 \times 10^3) - (1.333 \times 10^4)}{10^7 \left( \frac{1}{10} + \frac{1}{5} \right)} = 6.6667 \times 10^{-3}
\]

Iteration 1: \( i = 1 \)

\[
2u^{(1)} = 2u^{(0)} + \Delta u^{(1)} = 1.3333 \times 10^{-2}
\]

\[
2\varepsilon_a^{(1)} = 1.3333 \times 10^{-3} < \varepsilon_Y \quad \text{(elastic section!)}
\]

\[
2\varepsilon_b^{(1)} = 2.6667 \times 10^{-3} > \varepsilon_Y \quad \text{(plastic section!)}
\]

\[
1F_a^{(1)} = 1.3333 \times 10^4; \quad 1F_b^{(1)} = (ET(2\varepsilon_b^{(1)} - \varepsilon_Y) + \sigma_Y)A = 2.0067 \times 10^4
\]

\[
(1K_a + 1K_b)\Delta u^{(2)} = 2R - 2F_a^{(1)} - 2F_b^{(1)} \Rightarrow \Delta u^{(2)} = 2.2 \times 10^{-3}
\]
Introduction to non-linear analysis

• We look at the example again – simple bar

<table>
<thead>
<tr>
<th>i</th>
<th>$\Delta u^{(i)}$</th>
<th>$u^{(i)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1.45E-03</td>
<td>1.55E-02</td>
</tr>
<tr>
<td>3</td>
<td>1.45E-03</td>
<td>1.70E-02</td>
</tr>
<tr>
<td>4</td>
<td>9.58E-04</td>
<td>1.79E-02</td>
</tr>
<tr>
<td>5</td>
<td>6.32E-04</td>
<td>1.86E-02</td>
</tr>
<tr>
<td>6</td>
<td>4.17E-04</td>
<td>1.90E-02</td>
</tr>
<tr>
<td>7</td>
<td>2.76E-04</td>
<td>1.93E-02</td>
</tr>
</tbody>
</table>
The continuum mechanics incremental equations

- The basic problem:

We want to establish the solution using an incremental formulation

The equilibrium must be established for the considered body in its current configuration

In proceeding we adopt a Lagrangian formulation where we track the movement of all particles of the body (located in a Cartesian coordinate system)

Another approach would be an Eulerian formulation where the motion of material through a stationary control volume is considered
The continuum mechanics incremental equations

- The basic problem:

\[ \delta \mathbf{u} = \begin{pmatrix} \delta u_1 \\ \delta u_2 \\ \delta u_3 \end{pmatrix} \]

Configuration corresponding to variation in displacements \( \delta \mathbf{u} \) at \( t + \Delta t \).

Configuration at time \( t \):
- Surface area \( S \)
- Volume \( V \)

Configuration at time \( t + \Delta t \):
- Surface area \( S + \Delta S \)
- Volume \( V + \Delta V \)

\( x_1 (\text{or} \quad 0 x_1, \quad t x_1, \quad t + \Delta t x_1) \)
The continuum mechanics incremental equations

• The Lagrangian formulation

We express equilibrium of the body at time \( t+\Delta t \) using the principle of virtual displacements

\[
\int_{t+\Delta \mathcal{V}} (t+\Delta t) \tau_{ij} \delta e_{ij} d(t+\Delta t) V = (t+\Delta t) \mathcal{R}
\]

\( t+\Delta t \tau \) : Cartesian components of the Cauchy stress tensor

\[ \delta e_{ij} = \frac{1}{2} \left( \frac{\partial \delta u_i}{\partial t+\Delta t x_j} + \frac{\partial \delta u_j}{\partial t+\Delta t x_i} \right) \] strain tensor corresponding to virtual displacements

\( \delta u_i \) : Components of virtual displacement vector imposed at time \( t + \Delta t \)

\( t+\Delta t x_i \) : Cartesian coordinate at time \( t + \Delta t \)

\( t+\Delta t V \) : Volume at time \( t + \Delta t \)

\[
(t+\Delta t) \mathcal{R} = \int_{t+\Delta \mathcal{V}} (t+\Delta t) f_i^B \delta u_i d(t+\Delta t) V + \int_{t+\Delta \mathcal{S}_f} (t+\Delta t) f_i^S \delta u_i^S d(t+\Delta t) S
\]
The continuum mechanics incremental equations

- The Lagrangian formulation

We express equilibrium of the body at time \( t + \Delta t \) using the principle of virtual displacements

\[
R = \int_{V}^{t+\Delta t} f_i^B \delta u_i dV + \int_{S_f}^{t+\Delta t} f_i^S \delta u_i^S dS + \Delta t \delta u
\]

where

- \( f_i^B \): externally applied forces per unit volume
- \( f_i^S \): externally applied surface tractions per unit surface
- \( S_f \): surface at time \( t + \Delta t \)
- \( \delta u_i^S \): \( \delta u_i \) evaluated at the surface \( t+\Delta t S_f \)
The continuum mechanics incremental equations

- The Lagrangian formulation

We recognize that our derivations from linear finite element theory are unchanged – but applied to the body in the configuration at time $t + \Delta t$
The continuum mechanics incremental equations

- In the further we introduce an appropriate notation:

Coordinates and displacements are related as:

\[ \dot{x}_i = \dot{0}x_i + \dot{u}_i \]
\[ \dot{t+\Delta t}x_i = \dot{0}x_i + \dot{t+\Delta t}u_i \]

Increments in displacements are related as:

\[ \dot{u}_i = \dot{t+\Delta t}u_i - \dot{t}u_i \]

Reference configurations are indexed as e.g.:

\[ \dot{t+\Delta t}0f_i^S \]
where the lower left index indicates the reference configuration

\[ \dot{t+\Delta t}t_{ij} = \dot{t+\Delta t}t_{ij} \]

Differentiation is indexed as:

\[ \dot{t+\Delta t}0u_{i,j} = \frac{\partial \dot{t+\Delta t}u_i}{\partial \dot{0}x_j} , \quad \dot{0}x_{m,n} = \frac{\partial \dot{0}x_m}{\partial \dot{t+\Delta t}x_n} \]