Traffic Theory
from First Principles

Self-Organized Transport Phenomena
in Physics, Traffic, Crowds,
and Economics

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Introduction

This book finally fulfils my old dream to write an English volume on transportation processes, after a German book entitled “Verkehrsdynamik”, which was published in 1997. Compared to the German version, this volume not only contains a summary of what I learned since I have been entering the field of traffic modeling back in 1995. It also follows a different concept in order to explain the new understanding that has been gained since the classical books on traffic flow theory, e.g. the ones by Gazis [1] or Leutzbach [2]. In particular, it attempts to highlight the greater generality and wider importance of findings in traffic theory.

Those readers who expect a complete review of the field of traffic and transport modeling shall be warned right away: This is rather a personal view of transport processes, oriented at the classification and understanding of different phenomena observed in a variety of transport systems. Rather than trying to present a complete list of traffic models published in the past, as I have attempted in my Review of Modern Physics article of 2001 [3], this time the focus will be on a consistent explanation of the different self-organization and pattern-formation phenomena that are observed in transport systems. These phenomena are often surprising, at times even counter-intuitive.

The reason for this kind of presentation is two-fold: First of all, there already exists a significant list of excellent reviews of traffic modeling [3, 4, 5, 6]. Therefore, there is no need for another one. Second, transport phenomena play an important role in many different systems, not only traffic systems. This ranges from physics over chemistry and biology upto economics and even computer or social science. Examples include

- electrical currents and fluid flows,
- reaction-diffusion systems and flow reactors,
- cellular traffic and physiological transport,
- food chains and ecological systems,
- epidemic spreading and cascading effects,
- commodity and money flows,
While this list is far from being complete, we expect that there should be many similarities in the behavior of different transport phenomena. However, it is important to understand not only the commonalities, but also the differences. Therefore, the question is: “What are the possible phenomena observed in transport systems, and what are the mechanism behind them?” For example, one would like to know how traffic jams come about and what can be done to avoid them. Moreover, it is interesting to study why so many different systems from vehicle and pedestrian traffic, upto biological, economic, and engineered systems such as critical infrastructures (including electrical and information networks) have many features in common. Obviously, traffic flows, material flows, money flows, information flows, and work flows must be partially governed by common mathematical laws, despite all their differences.

This is the reason why we pursue an abstract, simplified approach to flows and transportation processes. It will focus on the essentials, i.e. the phenomena and mechanisms only. Another planned book volume will then attempt to present realistic models, which are more complicated in terms of the number of parameters and often not analytically tractable anymore. That volume shall present a problem-oriented approach in the sense that it asks: “What are the facts or observations and how could they be described?”

In many cases, there are several alternative descriptions of the same observation [?, ?]. It is a consequence of parameter-rich models that many different models can fit the data equally well. Therefore, it is often a matter of taste what models to prefer. Some of them may be better motivated, easier to handle, simulate, or calibrate, or theoretically more consistent than others. And some of them may actually fall into the same model class, i.e. may be mapped onto each other by means of mathematical transformations. In such a case, they would deliver qualitatively the same results, and there may only be a difference in the quantitative performance. For example, the simplified models in this book fall into the same model class as a considerable number of more realistic models. As a consequence, they already represent the same features and the same level of complexity as their more realistic “brothers” and “sisters”. Hence, these “toy models” represent the “stylized facts”, and they are suited to represent the model class.

Who should read this book? It is written to provide new insights for researchers in fields dealing with traffic, material, and other kinds of flows. Therefore, it is intended for researchers interested in transport phenomena, ranging from physicists over traffic scientists, economists, biologists, computer scientists, and engineers. But is is also written to give an in-depth introduction for graduate and postgraduate students. The book was actually the basis of a course on traffic dynamics and human coordination in space and time.
that I gave at ETH Zurich to a mixed audience composed of natural scientists and engineers from different study directions.

When I gave this course, a miracle occurred. The concept of the book changed considerably, and it gained in several ways: First of all, I have added many figures and additional motivation, and more detailed explanations. Second, I noticed that many properties of traffic flows could be analytically derived from much more fundamental principles than I was aware of before. Rather than a collection of previous publications, it became a derivation of traffic theory from first principles, with many new and original contributions. Third, I have decided not to assume that readers would know how to work with complex numbers, infinite series, differential equations etc. I have rather included a lot of exercises to introduce all these methods, and step-by-step solutions are given in the end of this book. That is, the book lifts all the methodological secrets that journal publications usually don’t disclose. This makes the book accessible to a non-specialist readership. Even undergraduate students with a background in mathematics could handle it.

In some way, this book has become an illustration of how linear algebra and analysis can be practically applied. Here, readers can see what all the methods taught in mathematical university courses are actually good for. Everything from the differentiation and integration rules of calculus, series expansions, complex numbers, ordinary and partial differential equations, eigenvalue problems and instability analysis can be exercised here. It’s actually very illuminating to see what all these methods are good for, and how much they can contribute to gain a fundamental understanding of the fascinating world of transport phenomena.

Finally, I would like to thank all the people who have contributed to the possibility of this book. Maybe I should start mentioning our former dean, Prof. Siegbert Liebig, who encouraged me to write a book on transportation in order to continue the tradition of the “Friedrich List” Faculty of Transportation and Traffic Sciences, which includes the work of Gerhart Potthoff [7]. I would like to thank my PhD and postdoc students Lubos Buzna, Anders Johansson, Arne Kesting, Stefan Lämmer, Seyyed Amin Mazloumian, Mehdi Moussaid, Martin Schönhof, and Martin Treiber, who have obtained many interesting results, prepared many illustrations. The list of people I have fruitfully collaborated with in the past 10 years is even longer, so that I cannot mention them all, but I am very grateful for the phantastic collaboration with them and suggest the reader to look up the references listed in the “Further Reading” sections. Peter Felten has prepared many additional illustrations. My personal assistant Dietmar Huber, my secretary Martina Seifert, and many team members have alleviated my work. And besides ETH Zurich, Dresden University of Technology and the Collegium Budapest – Institute of Advanced Study have provided me with stimulating environments and excellent working conditions.

This book has also profited from several funds and research projects by the German Research Foundation (DFG), by the Volkswagen AG within the
1. Introduction

BMBF project INVENT and other projects, by the Volkswagen and Daimler Benz Foundation, and by the “Cooperative Center for Communication Networks Data Analysis”, a NAP project sponsored by the Hungarian National Office of Research and Technology. In this connection, I would like to express my thanks to all sponsors and collaboration partners.

Having said this, there is no reason to wait any longer. Let’s jump into the subject right away!

1.1 Traffic, Crowds, Production and Logistics as Complex Systems

STILL TO BE WRITTEN

![Fig. 1.1. Measured trajectories of vehicles on a freeway which are passing a backwards propagating, localized area of congested traffic (from Coifman [?]).](realtraj)

1.2 Further Reading

2 Models for Interactive Motion

2.1 Introduction

This chapter serves the purpose of introducing a couple of simple models for transport processes of different kinds, which will later on be studied in more detail. The choice of models is not meant to be representative for the wide field of transportation models. I have rather decided to choose some models which are simple in terms of a small number of parameters and suited to cover different kinds of methods and application areas. This includes models for one-dimensional transport, two-dimensional motion, or flows in networks. It also covers microscopic and macroscopic, deterministic and stochastic models.

Keeping models simple implies that they are usually not detailed enough to describe realistic systems such as vehicle traffic or pedestrian crowds quantitatively well. Still, these models capture their basic features. They are also more suitable for an analytical study of the mechanisms underlying the phenomena observed. In fact, many transportation systems are based on non-linear interactions, which leads to many self-organized spatio-temporal pattern formation phenomena. Some of these are quite surprising or even counter-intuitive such as the freezing-by-heating or the slower-is-faster effects (see Secs. ?? and ??).

2.2 The Social Force Model

Most transport processes concern discrete units or elements $i$ such as particles, cars, people, packets, commodities, money, bits or bytes. Therefore, models for the behavior of these single elements are widespread. If these elements $i$ move continuously in space and $x_i(t)$ represents the location in that space at time $t$, we have the following equation of motion:

$$\frac{dx_i(t)}{dt} = \frac{dx_i(t)}{dt} = \mathbf{v}_i(t).$$  \hfill (2.1)
In other words, the change of location with time is given by the velocity \( \mathbf{v}_i(t) \). The term “speed” will be used for the absolute value \( v_i(t) = \|\mathbf{v}_i(t)\| \) of the velocity.

Changes of velocity \( d\mathbf{v}_i/dt \) are naturally described by an acceleration equation. In classical mechanics, this equation is typically of the form

\[
m_i \frac{d\mathbf{v}_i}{dt} = \mathbf{F}_i(t) = \mathbf{F}_i^0(t) + \sum_{j \neq i} \mathbf{F}_{ij}(t).
\]

Herein, \( m_i \) denotes the mass of a particle or solid body \( i \) and reflects its inertia. The potentially time-dependent vector \( \mathbf{F}_i^0(t) \) represents an external force (e.g. the force due to a gravitational field), while \( \mathbf{F}_{ij}(t) \) are pair interaction forces (e.g. mutual magnetic attraction or repulsion forces). For these pair interaction forces,

\[
\mathbf{F}_{ij}(t) = -\mathbf{F}_{ji}(t)
\]

applies, which is called Newton’s third law or the law of \( \text{actio} = \text{reactio} \). Accordingly, if a body \( j \) exerts a force \( \mathbf{F}_{ij} \) on body \( i \), this will exert a force of equal size, but opposite direction on body \( j \).

Let us divide all forces by the mass \( m_i \) and define \( \mathbf{f}_i(t) = \mathbf{F}_i(t)/m_i \), \( \mathbf{f}_i^0(t) = \mathbf{F}_i^0(t)/m_i \), as well as \( \mathbf{f}_{ij}(t) = \mathbf{F}_{ij}(t)/m_i \). Then, the resulting acceleration equation becomes

\[
\frac{d\mathbf{v}_i}{dt} = \mathbf{f}_i(t) = \mathbf{f}_i^0(t) + \sum_{j \neq i} \mathbf{f}_{ij}(t).
\]

Note that, now, we will usually not have a relationship of the kind \( \mathbf{f}_{ij} = -\mathbf{f}_{ji} \) anymore.

As it will simplify our further discussion, we will introduce some abbreviations, here: Let

\[
\mathbf{d}_{ij}(t) = \mathbf{x}_j(t) - \mathbf{x}_i(t)
\]

be the directional vector between elements \( i \) and \( j \), where \( \mathbf{x}_i(t) \) and \( \mathbf{x}_j(t) \) denote the locations of their center of mass at time \( t \). Then,

\[
d_{ij}(t) = \|\mathbf{d}_{ij}(t)\| = \|\mathbf{x}_j(t) - \mathbf{x}_i(t)\|
\]

is the distance between the centers of mass of both elements. Moreover,

\[
\mathbf{e}_{ij}(t) = \frac{\mathbf{d}_{ij}(t)}{\|\mathbf{d}_{ij}(t)\|} = \frac{\mathbf{x}_j(t) - \mathbf{x}_i(t)}{\|\mathbf{x}_j(t) - \mathbf{x}_i(t)\|}
\]

represents the directional unit vector pointing from element \( i \) to \( j \). With this, we will turn to the example of pedestrian motion for illustration, although the above equations can be applied to the interaction of planets, gas or fluid particles, vehicles, or even biological cells as well [1, 2].

If we want to describe pedestrian motion (see Table 2.1), the different effects on their acceleration behavior do normally not have a direct impact.
2.2 The Social Force Model

<table>
<thead>
<tr>
<th>Stimulus</th>
<th>Simple/standard situations</th>
<th>Complex/new situations</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Response</strong></td>
<td>Automatic re-action, ‘reflex’</td>
<td>Result of evaluation, decision process</td>
</tr>
<tr>
<td><strong>Characterization</strong></td>
<td>Well predictable</td>
<td>Highly probabilistic</td>
</tr>
<tr>
<td><strong>Modeling concept</strong></td>
<td>Social force model, etc.</td>
<td>Decision theoretical model, etc.</td>
</tr>
<tr>
<td><strong>Example</strong></td>
<td>Pedestrian motion</td>
<td>Destination choice by pedestrians</td>
</tr>
</tbody>
</table>

Table 2.1. Classification of behaviors according to their complexity (after [3]).

on the body (if the density is not extremely high). Rather brain and body transform the information about the presence and motion of other people into own, self-driven actions (see Fig. 2.1). The related acceleration or deceleration forces are usually called “social forces”.

![Fig. 2.1. Schematic representation of processes leading to behavioral changes (after [3]).](image-url)
The social forces \( f_{ij}(t) \) are often of repulsive nature, reflecting territorial effects and the desire not to collide with someone else. One of the simplest specifications is

\[
f_{ij}^{\text{rep}}(d_{ij}(t)) = -A_i e^{-d_{ij}(t)/B_i} e_{ij}(t), \tag{2.8}
\]

where \( A_i \) reflects the interaction strength and \( B_i \) the interaction range, assuming an exponential decay of the interaction force with distance. For improved specifications see Refs. [3, 4, 5].

When the common motion and clustering of groups of friends or family members shall be described as well, additional attractive social forces come into play. A particularly simple specification is

\[
f_{ij}^{\text{att}}(d_{ij}(t)) = C_{ij} d_{ij}(t) e_{ij}(t) = C_{ij} d_{ij}(t) \tag{2.9}
\]

(see Fig. 2.2), where \( C_{ij} = 0 \), if \( i \) and \( j \) do not belong to the same group. Altogether we have

\[
f_{ij}(t) = f_{ij}^{\text{rep}}(d_{ij}(t)) + f_{ij}^{\text{att}}(d_{ij}(t))
= [C_{ij} d_{ij}(t) - A_i e^{-d_{ij}(t)/B_i}] e_{ij}(t). \tag{2.10}
\]

---

**Fig. 2.2.** Illustration of Eq. (2.10), i.e. of the combined effect of weak long-term attraction and strong short-term repulsion between two pedestrians who like to walk together (such as family members or friends). A force of strength zero (dotted line) corresponds to the typically resulting distance between two pedestrians walking together.
The force $f^0_i(t)$ splits up into several contributions:

$$f^0_i(t) = \frac{v^0_i e^0_i(t) - v_i(t)}{\tau_i} + f^b_i(t) + \xi_i(t). \quad (2.11)$$

The first term on the right-hand side reflects an exponential adaptation of the actual velocity $v_i(t)$ to the desired speed $v^0_i$ and the desired direction $e^0_i$ of motion, with a “relaxation time” of $\tau_i$. The second term describes repulsive interactions with boundaries like streets or walls. If $d_b(t)$ denotes the smallest distance to the boundary and $e_b(t)$ the direction between pedestrian $i$ and the closest point of the boundary, we may assume a relationship of the form

$$f^b_i(t) = -A^b_i e^{-d_b(t)/B^b_i} e_b(t). \quad (2.12)$$

$A^b_i$ and $B^b_i$ are again individual parameters reflecting the strength and range of the exponentially decaying, repulsive interaction. Finally, $\xi_i(t)$ are fluctuating forces intended to represent unsystematic variations in individual behavior. In other words, for the average value of the fluctuation force we assume

$$\langle \xi_i(t) \rangle = 0. \quad (2.13)$$

For the correlations between the fluctuations forces, one may consider different possible specifications. The simplest is certainly a Gaussian white noise with

$$\langle \xi_i(t) \xi_j(t + \tau) \rangle = \delta_{ij} \delta(\tau), \quad (2.14)$$

which means that the fluctuating forces of different pedestrians are independent from each other and do not have a “memory effect”. (For details on white noise see Refs. [6].) Here

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \quad (2.15)$$

is the Kronecker function and $\delta(\tau)$ Dirac’s delta function with $\delta(\tau) = 0$ for $\tau \neq 0$ and

$$\int_{-\infty}^{t} d\tau \delta(\tau) = \begin{cases} 0 & \text{if } t < 0 \\ 1/2 & \text{if } t = 0 \\ 1 & \text{if } t > 0. \end{cases} \quad (2.17)$$

(For details regarding the delta function, see Ref. [7].)

In summary, the social force model of pedestrian dynamics reads

$$\frac{dv_i}{dt} = \frac{v^0_i e^0_i(t) - v_i(t)}{\tau_i} + f^b_i(t) + \sum_{j(\neq i)} f_{ij}(t) + \xi_i(t) \quad (2.18) \{ \text{sfmodel} \}$$

(see Fig. 2.3).

1. The delta function may be imagined here as a normal distribution

$$n(\tau) = \frac{1}{\sqrt{2\pi\theta}} e^{-\tau^2/(2\theta)} \quad (2.16)$$

in the limit $\theta \to 0$ of vanishing variance $\theta \geq 0$. 

### 2.2 The Social Force Model

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(see Fig. 2.3).

1. The delta function may be imagined here as a normal distribution

$$n(\tau) = \frac{1}{\sqrt{2\pi\theta}} e^{-\tau^2/(2\theta)} \quad (2.16)$$

in the limit $\theta \to 0$ of vanishing variance $\theta \geq 0$. 

Fig. 2.3. Illustration of the different “social forces” influencing the motion of a pedestrian.

2.3 Car-Following Models

Fig. 2.4. Animation of the dynamics of a car-following model (see http://www.traffic-simulation.de).

Another class of models that is related to social force models are car-following models (see Fig. 2.4). In this case, element $i$ corresponds to a single vehicle. According to the modeling philosophy of car-following (or follow-the-leader) models, the longitudinal dynamics of a vehicle along the road is determined by the distance
2.3 Car-Following Models

\[ d_i(t) = d_{i,i-1}(t) = x_{i-1}(t) - x_i(t) \]  \hspace{1cm} (2.19)

to the vehicle \( i-1 \) ahead (the “leader”), its own speed \( v_i(t) \), and potentially also by the relative speed

\[ \Delta v_i(t) = v_i(t) - v_{i-1}(t) = -\frac{dd_i(t)}{dt}. \]  \hspace{1cm} (2.20) \{e219\}

Analogously to the social force model for pedestrians, we may assume an acceleration equation of the form

\[ \frac{dv_i}{dt} = f_i(d_i(t), v_i(t), \Delta v_i(t)) = \frac{v_0 - v_i(t)}{\tau_i} + f_{i,i-1}(d_i(t), v_i(t), \Delta v_i(t)) + \xi_i(t), \]  \hspace{1cm} (2.21) \{cf\}

where \( f_i(d_i, v_i, \Delta v_i) \) represents the acceleration function and \( f_{i,i-1}(d_i, v_i, \Delta v_i) \leq 0 \) is basically a repulsive force in order to avoid accidents. For the moment, we will set \( \xi_i(t) = 0 \), while we will study the effect of fluctuations \( \xi_i(t) \) in Sec. ??.

\( v_0 \) represents again the free (“desired”) speed or speed limit, while \( \tau_i \) is the relaxation time determining the duration of acceleration processes. With the abbreviation

\[ v_e(d, v, \Delta v) = v_0 + \tau_i f_{i,i-1}(d_i, v_i, \Delta v_i) \leq v_0, \]  \hspace{1cm} (2.22) \{e221\}

the car-following model given by Eqs. (2.20) to (2.22) can be rewritten as

\[ \frac{dv_i}{dt} = v_e(d_i(t), v_i(t), \Delta v_i(t)) - v_i(t) / \tau_i. \]  \hspace{1cm} (2.23) \{erstmal\}

The so-called optimal velocity model (OVM) [8] makes the simple specification

\[ v_e(d, v, \Delta v) = v_0(d) = \frac{v_0}{2} \left[ \tanh \left( \frac{d-l}{s_0} - 2 \right) + \tanh(2) \right], \]  \hspace{1cm} (2.24) \{opvel\}

(see Fig. 2.5), where \( l \) represents the vehicle length and \( s_0 > 0 \) is a suitable parameter value. The related acceleration equation reads:

\[ \frac{dv_i(t)}{dt} = \frac{1}{\tau_i} \left[ v_e(d_i(t)) - v_i(t) \right] \]  \hspace{1cm} (2.25) \{OptVel\}

and can be easily solved together with the equation

\[ \frac{dd_i}{dt} = v_{i-1}(t) - v_i(t). \]  \hspace{1cm} (2.26) \{toge\}

Of course, there are many other specifications [9, 10, 11], particularly as the “relaxation time” \( \tau \) in the OVM must be chosen unrealistically small, if accidents shall be avoided [?]. An additional dependence on the relative velocity \( \Delta v \) allows for a much smoother driving behavior. Nevertheless, the optimal velocity model is still a very useful toy model to understand basic features of traffic instability, as we shall see in Sec. 5.2) and ??.
2.4 Macroscopic Dynamics and Continuity Equation

The social force model is an individual-based approach, which allows one to study detailed interaction effects and the consequences of heterogeneity, i.e. effects of differences in the behaviors of single elements. This is particularly relevant for understanding systems involving humans. In many cases, however, it is not possible to calculate or evaluate the behavior of all elements of a large system, just because that would require too much computer power. And even if it was possible to calculate the behavior of all elements, it would not make sense to do it, as this information would usually not be needed.

Most of the time, it is sufficient to work with “macroscopic” models. These models focus on the dynamics of the density $\rho(x, t)$ of elements in space $x$ and time $t$ and the average velocity $V(x, t)$, sometimes also on further quantities such as the variance of velocity. One advantage of such models is their greater suitability for analytical investigations. For example, we will later on determine stability conditions for macroscopic equations (see Chap. ??).

In many cases, the macroscopic equation for the density is given by the continuity equation. For simplicity, we will introduce it for the one-dimensional case used to describe the vehicle flow along a freeway, although the continuity equation can be formulated for $n$ dimensions. It results from the conservation of the number of elements (vehicles). If this conservation is not fulfilled, the continuity equation must be modified. In the following, however, we will
assume conservation. Then, the one-dimensional continuity equation has the form
\[
\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial}{\partial x} \left[ \rho(x,t)V(x,t) \right] = \text{source terms} - \text{sink terms},
\] (2.27)
where \( x \) is the location along the freeway and \( t \) the time. In contrast to the total time derivative
\[
\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \frac{dx}{dt} = \frac{\partial}{\partial t} + V \frac{\partial}{\partial x},
\] (2.28)
the partial derivatives \( \partial/\partial t \) and \( \partial/\partial x \) are used to indicate the derivation with respect to a single variable (here: \( t \) or \( x \)), if a function depends on several variables.

In a closed system, the source and sink terms vanish. Let us assume this system is a circular road of length \( L \). Integrating the density \( \rho(x,t) \) over this length gives the total number \( N(t) \) of elements:
\[
N(t) = \int_0^L dx \rho(x,t).
\] (2.29)

In a closed system, this number should be constant in time, i.e. \( dN(t)/dt = 0 \). It is easy to check this by integrating up the continuity equation (2.27). As the periodicity of the road implies \( \rho(L,t) = \rho(0,t) \) and \( V(L,t) = V(0,t) \), we get
\[
\frac{dN}{dt} = \int_0^L dx \frac{\partial \rho(x,t)}{\partial t} = -\int_0^L dx \frac{\partial}{\partial x} \left[ \rho(x,t)V(x,t) \right] \]
(2.30)
\[
= -[\rho(L,t)V(L,t) - \rho(0,t)V(0,t)] = 0,
\] (2.31)
as expected.

In summary, the continuity equation fulfils the requirement of conserving the number of elements (particles, pedestrians, vehicles, etc.) in a closed system. Now, we want to show the opposite, namely that the continuity equation follows from local vehicle conservation in each single location \( x \) of the system. For this, let us study an interval of length \( \Delta x \) with the location \( x \) in its center. The change in the number of entities located in the interval of length \( \Delta x \),
\[
\frac{dN(x,t)}{dt} = \int_{x-\Delta x/2}^{x+\Delta x/2} dx \frac{\partial \rho(x,t)}{\partial t} \approx \Delta x \frac{\partial \rho(x,t)}{\partial t},
\] (2.32)
is given by the number of entities flowing into this interval via its boundaries, minus the number of entities flowing out. Starting with Eq. (2.31), this gives
Here, the quantity
\[ Q(x, t) = \rho(x, t)V(x, t) \] (2.34)
represents the flow. The approximate relationship (2.33) becomes exact in the limit \( \Delta x \to 0 \). It directly implies the continuity equation
\[ \frac{\partial \rho(x, t)}{\partial t} + \frac{\partial}{\partial x} \left[ \rho(x, t) V(x, t) \right] = 0, \] (2.35)
which can be rewritten as
\[ \frac{d\rho(x, t)}{dt} = \frac{\partial \rho(x, t)}{\partial t} + V(x, t) \frac{\partial \rho(x, t)}{\partial x} = -\rho(x, t) \frac{\partial V(x, t)}{\partial x} \] (2.36)
by application of the product rule of Calculus [12],
\[ \frac{d}{dx} [u(x)v(x)] = u(x) \frac{dv(x)}{dx} + \frac{du(x)}{dx} v(x). \] (2.37)
\( d\rho/dt \) is called the “total” or “substantial” time derivative of the density, see Eq. (2.28), and \( \partial \rho(x, t)/\partial x \) denotes the spatial gradient of the density. The total time derivative describes the change of of the density in time from the perspective of an element moving with velocity \( V(x, t) \) (see Fig. ??), while the partial time derivative \( \partial \rho/\partial t \) describes the density change at the fixed location \( x \). Equation (2.36) shows that the density (as long as it changes continuously in time) can never become negative, as it would become zero before, which would simultaneously cause the substantial time derivative of the density to become zero.

Note that the continuity equation is absolutely fundamental, and it has a great range of validity. Therefore, we will finally derive it directly from the equation of motion \( dx_i/dt = v_i \), using a trick that I learned from Isaac Goldhirsch. We represent the location \( x_i(t) \) of an element \( i \) in space by a delta function \( \delta(x - x_i(t)) \) and introduce some normalized, symmetrical smoothing function
\[ s(x' - x) = s(|x' - x|) = s(x - x') \] (2.38)
with
\[ \int_{-\infty}^{\infty} dx' s(x' - x) = 1 \] (2.39)
(for example, the normal distribution or a suitable triangular function). With this, we define the local density
\[ \rho(x, t) = \int_{-\infty}^{\infty} dx' s(x' - x) \sum_i \delta(x' - x_i(t)) \quad (2.40) \]
\[ = \sum_i s(x_i(t) - x). \quad (2.41) \]

Herein, we sum up over all particles \( i \). Furthermore, we define the average velocity \( V(x, t) \) as usual via a weighted average [with the weight function \( \delta(x' - x_i(t))s(x' - x) \)]:

\[ V(x, t) = \frac{\int_{-\infty}^{\infty} dx' \sum_i v_i(t)\delta(x' - x_i(t))s(x' - x)}{\int_{-\infty}^{\infty} dx' \sum_i \delta(x - x_i(t))s(x' - x)} \]
\[ = \frac{\int_{-\infty}^{\infty} dx' \sum_i v_i(t)\delta(x' - x_i(t))s(x' - x)}{\rho(x, t)} \]
\[ = \frac{\sum_i v_i(t)s(x_i(t) - x)}{\rho(x, t)}. \quad (2.42) \]

This implies

\[ Q(x, t) = \rho(x, t)V(x, t) = \sum_i v_i(t)s(x_i(t) - x) \quad (2.43) \]

and, therefore, the fluid-dynamic flow formula (2.34). Equations (2.41) and (2.42) show that the above derivation is related to the approach of smoothed particle hydrodynamics [13], where an element \( i \) is represented by a smooth function \( s(x - x_i(t)) \) rather than a delta function \( \delta(x - x_i(t)) \). Differentiation of Eq. (2.40) with respect to time and application of the chain rule gives

\[ \frac{\partial \rho(x, t)}{\partial t} = \int_{-\infty}^{\infty} dx' \sum_i \left( -\frac{dx_i}{dt} \right) \cdot \frac{\partial}{\partial x'} \delta(x' - x_i(t)) s(x' - x) \]
\[ = \int_{-\infty}^{\infty} dx' \sum_i v_i(t) \delta(x' - x_i(t)) \left[ \frac{\partial}{\partial x'} s(x' - x) \right], \quad (2.44) \]
where we have applied \( \frac{dx_i}{dt} = v_i(t) \) and partial integration to obtain the last results (see Exercise ??). That is, we have used the theorem

\[
\int_{-\infty}^{\infty} dx' \left[ \frac{\partial}{\partial x'} u(x') \right] v(x') = \left[ u(x) v(x) \right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dx' u(x') \left[ \frac{\partial}{\partial x'} v(x') \right] (2.45)
\]

and considered the vanishing of the first term after the equality sign due to the vanishing of \( u(x) v(x) \) at the boundaries. Taking into account the symmetry of the smoothing function \( s(x' - x) \), see Eq. (2.38), we may replace \( \partial s(x' - x)/\partial x' \) by \( -\partial s(x' - x)/\partial x \), which finally gives

\[
\frac{\partial \rho(x,t)}{\partial t} = -\frac{\partial}{\partial x} \int_{-\infty}^{\infty} dx' \sum_i v_i(t) \delta(x' - x_i(t)) s(x' - x) = -\frac{\partial}{\partial x} \left[ \rho(x,t) V(x,t) \right]. (2.46)
\]

To obtain this desired result, we have finally applied the definition (2.42) of the average velocity \( V(x,t) \). As a consequence of this, the validity of the continuity equation does not require an averaging over large numbers of entities, i.e. macroscopic volumes to average over. This makes the equation so powerful.

### 2.5 Lighthill-Whitham-Richards Model (LWR Model)

We will now have to address the question, how to specify the relationship for the average velocity \( V(x,t) \). The specification depends very much on the system considered and can vary significantly. That is, the underlying acceleration equation can make a big difference, which is not surprising.

In the case of freeway traffic, it is common to apply the continuity equation in one spatial dimension, namely the coordinate \( x \) along the freeway. In the absence of any on- or off-ramps, the continuity equation reads

\[
\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial Q(x,t)}{\partial x} = 0 \quad (2.47)
\]

The probably simplest and most influential specification of the traffic flow \( Q(x,t) \) has been independently proposed by Lighthill and Whitham on the one hand [14] and Richards [15] on the other hand. Their fluid-dynamic traffic model assumes that the vehicle flow \( Q(x,t) \) is just a function of the density \( \rho(x,t) \):

\[
Q(x,t) = Q_e(\rho(x,t)) \quad (2.48)
\]

The flow-density relationship \( Q_e(\rho) \) is often called the “fundamental diagram”, because it allows one to derive many fundamental properties by simple geometric construction (see Sec. ??). Inserting it into the continuity equation results in the Lighthill-Whitham-Richards model (LWR model)
\[ \frac{\partial \rho(x,t)}{\partial t} + \frac{\partial Q(x,t)}{\partial x} = \frac{\partial \rho(x,t)}{\partial t} + \frac{dQ_e(\rho)}{d\rho} \frac{\partial \rho(x,t)}{\partial x} = 0. \] (2.49)

Moreover, by defining the speed-density relationship
\[ V_e(\rho) = \frac{Q_e(\rho)}{\rho} \] (2.50)
(see Fig. 2.7) and the propagation speed of kinematic waves,
\[ c(\rho) = \frac{dQ_e(\rho)}{d\rho} = V_e(\rho) + \rho \frac{dV_e(\rho)}{d\rho}, \] (2.51) \{crho\}
we can rewrite the LWR model:
\[ \frac{\partial \rho(x,t)}{\partial t} + c(\rho) \frac{\partial \rho(x,t)}{\partial x} = 0. \] (2.52) \{s\}

As the average velocity is normally assumed to obey a monotonously decreasing relationship of the density (i.e. \( dV_e(\rho)/d\rho \leq 0 \)), we have \( c(\rho) \leq V_e(\rho) \). Moreover, the relationship \( Q_e(\rho) = \rho V_e(\rho) \) implies that the flow increases with the density as long as the density is small, while it tends to decrease at high densities due to the monotonously decreasing speed-density relationship.\(^2\)

\( \text{Fig. 2.7. Illustration of typical speed-density and flow-density relationships for a multi-lane freeway. The flow-density relationship is also known as fundamental diagram. Note that empirical data of the flow are widely scattered in the congested density regime, where } dQ_e/d\rho \leq 0. \text{ An explanation for this will be offered in Sec. ?? INSERT NEW FIGURES ??} \)

\(^2\) Note, however, that exceptions are possible, i.e. for special speed-density relationships, the fundamental diagram can have several maxima [?] [?].
For simplicity and further use in Sec. (11.1), let us assume the example of a simple triangular fundamental diagram, which is defined by

\[
Q_e(\rho) = \begin{cases} 
\rho V^0 & \text{if } 0 \leq \rho < \rho_c \\
\frac{1}{T} \left(1 - \frac{\rho}{\rho_{jam}}\right) & \text{if } \rho_c \leq \rho \leq \rho_{jam},
\end{cases}
\] (2.53) \{triangle\}

where \( \frac{1}{\rho_c} = \frac{1}{\rho_{jam}} + TV^0 \) (2.54) defines the density \( \rho_c \), at which a transition from free flow with \( dQ_e/d\rho > 0 \) to congested flow with \( dQ_e/d\rho < 0 \) takes place. \( V^0 \) has the meaning of the free (“desired”) speed or the speed limit, \( T \) is the safe time gap between successive vehicles, and \( \rho_{jam} \), the maximum or jam density.

According to Eq. (2.51) and with the abbreviation \( c_0 = -\frac{1}{\rho_{jam}T} \), (2.55) we have

\[
c(\rho) = \begin{cases} 
V^0 & \text{if } \rho < \rho_c, \\
c_0 & \text{if } \rho \geq \rho_c.
\end{cases}
\] (2.56)

The free speed determines the dynamics in the free flow regime, and for \( x \geq x_0 \) we have

\[
\rho(x, t) = \rho\left(x_0, t - (x - x_0)/V^0\right),
\] (2.57)
as can be easily checked by inserting this solution into equation (2.52) (see Exercise ??). In other words, the time-dependent density profile at \( x_0 \) propagates forward at the constant speed \( V^0 \) together with the vehicles. Throughout the congested regime, however, we have the relationship

\[
\rho(x, t) = \rho\left(x_1, t - (x_1 - x)/|c_0|\right)
\] (2.58)
with \( x \leq x_1 \). That is, density variations at \( x_1 \) propagate upstream at the speed \( c_0 \), which corresponds to the dissolution speed of congested traffic (i.e. to the propagation speed of the downstream jam front, after a traffic light has turned green).

In the mathematical theory of partial differential equations, \( V^0 \) and \( c_0 \) are called characteristic speeds \[16\]. Their existence has interesting implications for computer simulations, as the dynamics on a road section of homogeneous flow capacity between \( x = x_0 \) and \( x = x_1 \) is completely determined by the initial condition \( \rho(x, 0) \) with \( x_0 \leq x \leq x_1 \) and (primarily) the boundary conditions \( \rho(x_0, t), \rho(x_1, t) \). While the upstream boundary condition \( \rho(x_0, t) \) determines the area of free traffic flow, the downstream boundary condition \( \rho(x_1, t) \) determines the area of congested traffic flow. The same is true for other fundamental diagrams with one maximum \[17\].
For such fundamental diagrams, we will see in Secs. 11.1 and 3.3 that free and congested flow are separated by one shock front, the speed of which can also be determined from the traffic dynamics at the boundaries. Note that it is possible to have several congested areas, if the road consists of several sections with different flow capacities, i.e. if it contains several bottlenecks due to ramps or a varying number of lanes etc. (or if the fundamental diagram has several maxima [18, 19]). Therefore, it is reasonable to subdivide the road system into sections of homogeneous capacity and look for equations determining the dynamics of them as a function of the respective boundary conditions. This will be done in Sec. 11.1.

2.6 Second-Order Models

With Eq. (2.48), the Lighthill-Whitham-Richard model implicitly assumes an instantaneous adaptation of the vehicle flow and the average velocity to the equilibrium values $Q_e(\rho(x, t))$ and $V_e(\rho(x, t))$, respectively. As this is certainly an idealization, Whitham himself came up with a generalization, introducing a dynamic velocity equation [17]. In the following, we present a generalized form of his equation, which allows us to summarize the most prominent models proposed in the past:

$$\frac{\partial V}{\partial t} + V \frac{\partial V}{\partial x} + \frac{1}{\rho} \frac{\partial P}{\partial x} = \frac{1}{\tau} \left[ V_e(\rho) - V \right].$$

(2.59) {geschwin}

This equation is called the velocity equation and has a formal relationship with the Euler and Navier-Stokes equations from hydrodynamics [20]. The term $V \partial V / \partial x$ is referred to as transport or convection term. It describes a motion of the velocity profile with the vehicles. The pressure term reflects the dispersion effect of different vehicle speeds, i.e. a finite velocity variance (see Fig. 2.8). Other model specifications have rather the interpretation of an anticipation term, describing the reaction of identical driver-vehicle units to the traffic situation in their surrounding, in particular in front of them. The relaxation term $(V_e(\rho) - V) / \tau$ delineates the adaptation of the average velocity $V(x, t)$ to the density-dependent equilibrium velocity $V_e(\rho)$. This adaptation process is exponential in time with a relaxation time of $\tau$.

The main difference between the various macroscopic traffic models is their respective specification of the traffic pressure $P$, the relaxation time $\tau$, and the dynamic equilibrium velocity $V_e$. For example, Payne’s [21] and Papageorgiou’s [22] model is obtained for

$$P(\rho) = \frac{V_0 - V_e(\rho)}{2\tau},$$

(2.60) {prpayne}

where $V_0 = V_e(0)$ is the “free” or “desired” average velocity. In the limit $\tau \rightarrow 0$, Eqs. (2.59) and (2.60) imply
Fig. 2.8. Illustration of the effect of the pressure term. 

\[ V(x, t) = V_e(\rho(x, t)) - \frac{\tau}{\rho} \frac{\partial P}{\partial x} = V_e(\rho(x, t)) + \frac{1}{2\rho} \frac{dV_e(\rho)}{d\rho} \frac{\partial \rho}{\partial x}. \]  

(2.61)

Inserting this into the continuity equation (2.47) and defining the density-dependent "diffusion function"

\[ D(\rho) = \tau \frac{dP(\rho)}{d\rho} = -\frac{1}{2} \frac{dV_e(\rho)}{d\rho} \geq 0 \]  

(2.62)

{Paynepress}
results in

\[ \frac{\partial \rho(x, t)}{\partial t} + \frac{\partial Q(x, t)}{\partial x} = \frac{\partial}{\partial x} \left( D(\rho(x, t)) \frac{\partial \rho(x, t)}{\partial x} \right). \]  

(2.63)

{would}

Cremer multiplied the anticipation term by a factor \( \rho/(\rho + \kappa) \) with a positive constant \( \kappa \), in order to produce a more realistic behavior at small densities. Therefore, his model is characterized by the pressure gradient

\[ \frac{dP}{d\rho} = -\frac{\rho}{2\tau(\rho + \kappa)} \frac{dV_e(\rho)}{d\rho}. \]  

(2.64)

In the limit \( \tau \to 0 \), it results in Eq. (2.63) as well, but the diffusion function becomes

\[ D(\rho) = -\frac{1}{2(\rho + \kappa)} \frac{dV_e(\rho)}{d\rho} \geq 0. \]  

(2.65)
If \( D(\rho) \) is constant, i.e. in the special case of a linear velocity-density relation \( V_e(\rho) = V^0(1 - \rho/\rho_{\text{max}}) \), Eq. (2.63) agrees with the so-called Burgers equation

\[
\frac{\partial \rho(x, t)}{\partial t} + \frac{\partial Q(x, t)}{\partial x} = D \frac{\partial^2 \rho(x, t)}{\partial x^2}.
\]

The diffusion term \( D \frac{\partial^2 \rho}{\partial x^2} \) reflects an adjustment of vehicle speeds to the neighboring traffic situation, i.e. a reaction not only to the local density \( \rho \), but also to a change \( \partial \rho / \partial x \) of the density along the freeway. In Chap. 4, we will see that the diffusion term counteracts extrema in the density and, more generally, leads to a smoothing of the density profile.

The smoothing effect disappears in the limit \( D \to 0 \), which corresponds to the Lighthill-Whitham-Richard model. However, the LWR model also results in the limit \( \tau \to 0 \) from all macroscopic traffic models of the above class, in which the traffic pressure is not proportional to \( 1/\tau \). For example, in the model of Phillips [23] we have \( P = \rho \theta \), where \( \theta \) denotes the velocity variance. Moreover, the model of Kühne [24], Kerner and Konhäuser [25], and Lee et al. [26] (KKKL model) results for

\[
P = \rho \theta_0 - \eta_0 \frac{\partial V}{\partial x},
\]

where \( \theta_0 > 0 \) is a constant and \( \eta_0 > 0 \) a viscosity coefficient. The resulting Navier-Stokes-like traffic equation is

\[
\frac{\partial V}{\partial t} + V \frac{\partial V}{\partial x} = -\frac{\theta_0}{\rho} \frac{\partial \rho}{\partial x} + \frac{\eta_0 \partial^2 V}{\rho \partial x^2} + \frac{1}{\tau} [V_e(\rho) - V].
\]

In comparison with the similar model by Whitham [17], the contribution \( -\eta_0 \partial V / \partial x \) in Eq. (2.67) implies the additional viscosity term \( (\eta_0 / \rho) \partial^2 V / \partial x^2 \), as in the Navier-Stokes equation of hydrodynamics [20]. This has a similar smoothing effect as the diffusion term, but for the average velocity rather than the density.

Note that, in the following, it turns out that the traffic pressure \( P \) is not only dependent on the vehicle density \( \rho(x, t) \), but also on the average velocity \( V(x, t) \). Details can be found in Eq. (2.84) and Exercise ??.

According to the considerations over there, we can assume \( \partial P / \partial V < 0 \).

### 2.7 Micro-Macro Link

The large variety of macroscopic traffic models indicates that it is difficult to find the right specification of the traffic pressure and the speed-density relationship. Therefore, people had great difficulties to come up with micro- and macromodels that would produce identical simulation results. Even worse, Daganzo showed that more or less all macroscopic traffic models in the 1990ies
had serious theoretical inconsistencies [27]. It is, therefore, an interesting question, how follow-the-leader models are related to macroscopic models and vice versa. Is it possible to systematically derive a macroscopic traffic model from a microscopic one, as we derived the continuity equation from the equation of motion in Sec. ....? Well, in principle, the answer is “yes”, but the derivations are not trivial and involve some tricky approximations in order to close the system of equations.

One way to get from microscopic to macroscopic models is to derive gas-kinetic traffic equations [?, ?]. Besides the continuity equation for the density, these contain an equation for the velocity distribution \( P(v; x, t) \) as a function of space and time. This is determined via a gas-kinetic equation, which allows one to derive a macroscopic equation for the average velocity by application of an averaging procedure and certain approximations. An example of a macroscopic model determined in this way is the non-local, gas-kinetic-based traffic model (GKT model) [28]. It can be also formulated for several interacting lanes and different driver-vehicle classes. Furthermore, it is possible to consider velocity correlations among vehicles and effects of the finite vehicle size at large densities [29].

Here, we will apply a simpler procedure to establish a micro-macro-link. For this, we will use an approach reminding of smooth particle hydrodynamics introduced in Sec. 2.4. While we provide the derivation here for traffic flows in one dimension, similar results can be obtained in the \( n \)-dimensional case. Starting from Eq. (2.42), by differentiation with respect to time we get by application of the product rule

\[
\frac{\partial (\rho V)}{\partial t} = \frac{\partial}{\partial t} \int \frac{dx'}{\sum_{i}} v_{i}(t) \delta \left( x' - x_{i}(t) \right) s(x' - x) = \int_{-\infty}^{\infty} \frac{dv_{i}}{dt} \delta \left( x' - x_{i}(t) \right) s(x' - x) + \int_{-\infty}^{\infty} \frac{v_{i}(t)}{x'} \delta \left( x' - x_{i}(t) \right) s(x' - x) \quad (2.69)
\]

Now, we make use of the relationships \( dx_{i}/dt = v_{i}(t) \) and \( dv_{i}/dt = f_{i}(t) \) (see Eq. (2.21), and carry out the same steps as in Eqs. (2.44) to (2.46). This eventually gives

\[
\frac{\partial (\rho V)}{\partial t} = \int_{-\infty}^{\infty} \frac{dx'}{\sum_{i}} f_{i}(t) \delta \left( x' - x_{i}(t) \right) s(x' - x) + \frac{\partial}{\partial x} \int_{-\infty}^{\infty} \frac{dx'}{\sum_{i}} [v_{i}(t)]^{2} \delta \left( x' - x_{i}(t) \right) s(x' - x) . \quad (2.70)
\]
Furthermore, we define the velocity variance similar to the average velocity as

$$\theta(x, t) = \frac{\int_{-\infty}^{\infty} dx' \sum_i (v_i(t) - V(x,t))^2 \delta(x' - x_i(t)) s(x' - x)}{\rho(x, t)} = \sum_i [\delta v_i(t)]^2 s(x_i(t) - x) \rho(x, t), \quad (2.71)$$

where $\delta v_i(t) = v_i(t) - V(x, t)$. As the average value of $\delta v_i(t)$ is zero by definition (2.59) of the average velocity $V(x, t)$, we eventually end up with

$$\frac{\partial (\rho V)}{\partial t} + \frac{\partial}{\partial x} \left\{ \rho(x, t) [V(x, t)^2 + \theta(x, t)] \right\} = \sum_i f_i(t) s(x_i(t) - x). \quad (2.72)$$

Next, we apply the product rule and insert the continuity equation (2.35) for $\partial \rho/\partial t$. With the pressure definition

$$P(x, t) = \rho(x, t) \theta(x, t), \quad (2.73) \quad \{\text{pressure form}\}$$

this gives

$$\frac{\partial V(x, t)}{\partial t} + V(x, t) \frac{\partial V(x, t)}{\partial x} + \frac{1}{\rho(x, t)} \frac{\partial P(x, t)}{\partial x} = \frac{1}{\rho(x, t)} \sum_i f_i(t) s(x_i(t) - x). \quad (2.74)$$

For $f_i(t)$, we may now insert Eq. (2.21). For $v_i^0 = V^0$ we obtain

$$\frac{\partial V(x, t)}{\partial t} + V(x, t) \frac{\partial V(x, t)}{\partial x} + \frac{1}{\rho(x, t)} \frac{\partial P(x, t)}{\partial x} = \frac{V^0 - V(x, t)}{\tau} + \frac{1}{\rho(x, t)} \sum_i s(x_i(t) - x) f_{i,i-1}(t). \quad (2.75)$$

The last term in this equation describes the average acceleration of vehicles $i$ in the neighborhood of location $x$ due to interactions with the respective leading vehicles $i - 1$. For identical vehicles it may be further simplified, considering that $f_{i,i-1}(t) = f(x_{i-1}(t) - x_i(t))$, where $f$ is the distance-dependent interaction force.

For further simplification, let us now specify the smoothing function by the rectangular function

$$s(x_i - x) = \frac{\varrho}{2} \left\{ \begin{array}{ll} 1 & \text{if } |x_i - x| \leq 1/\varrho \\ 0 & \text{otherwise}, \end{array} \right. \quad (2.76)$$

with a large enough smoothing window of length $\Delta x = 2/\varrho$. Then, the number of vehicles $i$ within the smoothing interval $[x - 1/\varrho, x + 1/\varrho]$ is expected to be
\[ \rho \Delta x = 2\rho / \varrho, \text{ where } \rho \text{ represents the average vehicle density in this interval.} \]

Therefore,
\[ \rho(x, t) = \sum_i s(x_i(t) - x) = \frac{2\rho \varrho}{\varrho} = \rho, \quad (2.77) \]

which shows the consistency of this approach.

If the smoothing parameter \( \varrho \) is specified via the inverse vehicle distance
\[ \varrho = \varrho_k = \frac{1}{d_k} = \frac{1}{x_{k-1} - x_k} = \rho(x, t) \quad \text{for} \quad x_k < x \leq x_{k-1}, \quad (2.78) \]

the smoothing window of length \( \Delta x = 2/\varrho \) will usually contain only two vehicles \( k - 1 \) and \( k \) with \( x_k \leq x \leq x_{k-1} \). With this, the sum over \( i \) reduces to two terms with \( i = k \) and \( i = k - 1 \) only. This finally yields
\[
V(x, t) = \sum_i v_i(t)(x_i(t) - x) = v_k(t)s(x_k(t) - x) + v_{k-1}(t)s(x_{k-1}(t) - x)
\]
\[
= \frac{\varrho}{2} [v_{k-1}(t) + v_k(t)] = \rho(x, t)\frac{v_{k-1}(t) + v_k(t)}{2}
\]

(2.79)

and, considering Eq. (2.78),
\[
\sum_i s(x_i(t) - x)f(d_i(t)) = \frac{\varrho}{2} f(d_k) + \frac{\varrho}{2} f(d_{k-1}) = \frac{\varrho}{2} f\left(\frac{1}{\varrho_k}\right) + \frac{\varrho}{2} f\left(\frac{1}{\varrho_{k-1}}\right)
\]
\[
= \frac{\rho(x, t)}{2} f\left(\frac{1}{\rho(x, t)}\right) + \frac{\rho(x, t)}{2} f\left(\frac{1}{\rho(x + 1/\rho, t)}\right).
\]

(2.80)

In summary, the macroscopic velocity equation related to the optimal velocity model corresponds to
\[
\frac{\partial V(x, t)}{\partial t} + V(x, t)\frac{\partial V(x, t)}{\partial x} = -\frac{1}{\rho(x, t)} \frac{\partial}{\partial x} [\rho(x, t)\vartheta(x, t)] + \frac{v^0 - V(x, t)}{\tau} + \frac{1}{2} f\left(\frac{1}{\rho(x, t)}\right) + \frac{1}{2} f\left(\frac{1}{\rho(x + 1/\rho, t)}\right). \quad (2.81)
\]

In zeroth order Taylor approximation \( f(1/\rho(x + 1/\rho, t)) \approx f(1/\rho(x, t)) \), i.e., assuming small average gradients in the density,\(^3\) Eq. (2.75) finally becomes
\[
\frac{\partial V(x, t)}{\partial t} + V(x, t)\frac{\partial V(x, t)}{\partial x} + \frac{1}{\rho(x, t)} \frac{\partial \vartheta(x, t)}{\partial x} \approx \frac{v^0 - V(x, t)}{\tau} + f(1/\rho(x, t)). \quad (2.82)
\]

\(^3\) One could write many chapters on this assumption, as deviations from it are obviously essential to understand phenomena like fluctuations and turbulence. While these are beyond the focus of this book, a refinement of Eq. (2.82) is discussed in Exercise ??.
Using the abbreviation
\[ V_e(\rho) = V^0 + \tau f \left( \frac{1}{\rho(x,t)} \right) \leq V^0, \quad (2.83) \]
and assuming repulsive forces with \( f(1/\rho) \leq 0 \), this corresponds exactly to the velocity equation (2.59). However, we are now capable of deriving better approximations (see Exercise ??). A further discussion of the micro-macro link for the example of the optimal-velocity model can be found in Ref. [30].

Moreover, compared to the previous phenomenological specifications presented in Sec. 2.6, we have learned how to specify a macroscopic model that is consistent with a given microscopic one, which has useful implications for combined micro-macro simulations (see below). Finally, we have seen that the pressure term reflects the density times the variance of speeds. For the variance, one can either derive another partial differential equation similar to the calculations in this section, or one can close the system of equations by a functional relation of the kind
\[ \theta(x,t) = \alpha(\rho(x,t)) \left[ V(x,t) \right]^2. \quad (2.84) \]
This specification reproduces empirical data well [29] (see Fig. 2.9), and it guarantees that the variance of velocities becomes zero whenever the average vehicle velocity vanishes.

**Fig. 2.9.** INSERT FIGURE Fit function \( \alpha(\rho) = A_0 + \Delta A \{ 1 + \tanh(\rho - \rho_c)/\Delta \rho \} \) of the scaled variance \( \theta(x,t)/[V(x,t)]^2 \) with parameters \( A_0 = 0.008, \Delta A = 0.015, \rho_c = 0.28 \rho_{max}, \) and \( \Delta \rho = 0.1 \rho_{max} \) [28]. For an explanation of the higher relative variance in the congested density regime see Ref. [?].

As the rectangular smoothing function is discontinuous and problematic in terms of differentiation, it would be preferable to check our results for another symmetrical function \( s(x_i - x) \). In fact, we obtain the same result (2.82) (in zeroth order Taylor approximation), if we use the triangular smoothing function
\[ s(x_i - x) = \max \left[ \varrho(1 - \varrho |x_i - x|), 0 \right] \quad (2.85) \]
and consider \( |x_{k-1} - x| + |x - x_k| = (x_{k-1} - x) + (x - x_k) = x_{k-1} - x_k \), by which we specify again the parameter \( 1/\varrho \) of the smoothing function. This implies
\[ \rho(x,t) = s(x_{k-1}(t) - x) + s(x_k(t) - x) = 2 \varrho - \varrho^2(x_{k-1} - x_k) = \varrho = \frac{1}{x_{k-1} - x_k}. \quad (2.86) \]
The above triangular smoothing improves the micro-macro link based on a linear interpolation approach [31] (see Exercise ??). In contrast to the latter, the
triangular smoothing is theoretically consistent with the continuity equation (see Sec. 2.4). Consequently, it conserves the number of vehicles, as expected. However, both approaches can be used to perform coupled micro-macro simulations. In Figs. 2.10 and 2.11, for example, certain freeway sections were simulated microscopically, while the remaining freeway stretch was simulated macroscopically.

The most difficult part in these simulations is the specification of the boundary conditions, which must take into account the different characteristic speeds in free and congested traffic flows [32]. This difficulty is particularly pronounced at the interface between the microscopic and the macroscopic simulation area. While macroscopic quantities may be easily determined from microscopic ones in the way described above (by aggregation/spatial smoothing), for combined micro- and macro-simulations of traffic flows it is necessary to generate microscopic vehicle data at a location \(x_0\) from macroscopic quantities. This is done as follows: Whenever the time integral of the flow,

\[
N(x_0, t) = \int_{t_0}^{t} dt' \rho(x_0, t')V(x_0, t'),
\]

reaches integer values (say, at the times \(t_i\) with \(i \in \{1, 2, \ldots\}\)), new vehicles \(i\) are inserted, and their velocities are set to

\[
v_i(t_i) = V(x_0, t_i) + \xi_i.
\]

Herein, the random variable \(\xi\) is Gaussian distributed with variance \(\theta(x_0, t)\). If the variance is not specified or known, one may assume homogeneous vehicle velocities corresponding to \(\theta(x_0, t) = 0\) and \(\xi_i = 0\).

Note that the above micro-macro link may be extended to interaction forces that do not only depend on the vehicle distance, but also on the relative speed (approaching rate) or the vehicle velocity itself (see Exercise ??). Furthermore, it should be pointed out that, in view of the large gradients of the density and average velocity in measured traffic data, it is not justified to replace non-localities by gradient expansions. It yields better and theoretically more consistent results to keep the non-localities implied by the micro-macro link in the resulting macroscopic traffic equations [?]. An example of a non-local traffic model is the gas-kinetic-based traffic model (GKT model) [31].

### 2.8 Exercises

1. In the simplest specification of the social-force model [see Eq. (2.8)], there are two parameters \(A\) and \(B\). Try to describe how each of these parameters affects the interactions between pedestrians.
2. Restricting the social force model on its right-hand side to the relaxation term (i.e. the velocity adaptation term), we have
Fig. 2.10. Results of computer simulations of a circular road, part of which was simulated microscopically (dark blue), while the rest of it was simulated macroscopically (light red). Whereas the upper picture shows the forward propagation of a decaying initial perturbation at small vehicle density, the lower one illustrates the development of a backward moving traffic jam at medium density (after [31]).
Models for Interactive Motion

\[ dv_i(t) \frac{dt}{\tau} = v_i^0 e_i^0 - v_i(t) \]  (2.89)

What is the dynamical, general solution of this linear ordinary differential equation?

3. Describe the physical meaning of the equilibrium velocity in the car-following model [see Eq. (2.23)], and derive from it a relationship for the stationary velocity for the case of \( N \) identical vehicles on a circular road of length \( L \). (Please also list the assumptions used in the derivation.)

4. Show explicitly that the density \( \rho(x, t) \) in the continuity equation (2.35) can never become negative for \( t \geq 0 \), if it is non-negative at time \( t = 0 \).

5. Derive from the continuity equation (2.35) a partial differential equation for the vehicle density \( \rho'(x, t) = \rho(x, t)/I(x) \) per lane, where \( I(x) \) denotes the location-dependent number of lanes along a freeway. Show that the resulting equation has the form of a continuity equation with an additional term. How can this term be interpreted? Make a similar derivation, if the effectively usable width \( I(x, t) \) of the road does not only depend on the location, but also on time (e.g. due to some temporary blockage of parts of the road). What kinds of situations could be described by the resulting equation?

6. Derive the formula
from the product rule of Calculus. Furthermore, derive the formula

$$\int_{0}^{L} dx \frac{du(x)}{dx} v(x) = \left[ u(x)v(x) \right]_{0}^{L} - \int_{0}^{L} dx u(x) \frac{dv(x)}{dx}$$

for partial integration in $n$ spatial dimensions with Gauss’ integral theorem

$$\int_{\mathcal{V}} d^{n}x \nabla_{x} u(x) \cdot v(x) = \oint_{\partial \mathcal{V}} dA \cdot [ u(x)v(x) ]$$

where $\mathbf{F}(x)$ is a smooth vectorial function, $\partial \mathcal{V}$ represents the surface of the volume $\mathcal{V}$, over which the volume integral $\int_{\mathcal{V}}$ extends, and $\oint_{\partial \mathcal{V}}$ is a surface integral, where $d\mathbf{A}(x)$ represents infinitesimal surface elements to be imagined as vectors of size $dA = \|d\mathbf{A}\|$ perpendicular to the surface $\partial \mathcal{V}$ at place $\mathbf{x}$, pointing outside of the volume $\mathcal{V}$. The nabla symbol $\nabla_{x} = (\partial/\partial x_{1}, \ldots, \partial/\partial x_{n})'$ is the del operator (a vector differential operator), and the divergence $\text{div}\mathbf{F}(x) = \nabla_{x} \cdot \mathbf{F}(x)$ of a vector function $\mathbf{F}(x)$ is defined by

$$\nabla_{x} \cdot \mathbf{F}(x) = \sum_{k=1}^{n} \frac{\partial F(x)}{\partial x_{k}}.$$  

Herein, $x_{k}$ are the coordinates of the location $\mathbf{x} = (x_{1}, \ldots, x_{k}, \ldots, x_{n})'$. The prime (') turns a row vector into a column vector (and vice versa).

7. Derive the continuity equation in the $n$-dimensional from the conservation of the number of elements, i.e. from the fact that the temporal change in the number of elements in an infinitesimal (hyper-)cube of volume $\Delta x^{n}$ is given by the inflows minus the outflows through its surface.

8. Derive the $n$-dimensional continuity equation for the density

$$\rho(\mathbf{x}, t) = \int_{\mathcal{V}} d^{n}x' \ s(\|\mathbf{x}' - \mathbf{x}\|) \sum_{i} \delta(\mathbf{x}' - \mathbf{x}_{i}(t))$$

from the social force model (2.18), using the symmetric smoothing function $s(\|\mathbf{x}_{i}(t) - \mathbf{x}\|)$.

9. Start with the continuity equation

$$\frac{\partial \rho(\mathbf{x}, t)}{\partial t} + \nabla_{x} \cdot [ \rho(\mathbf{x}, t) \mathbf{V}(\mathbf{x}, t) ] = 0$$

for $n$ dimensions and derive that the number of elements in a closed system of volume $\mathcal{V}$ is constant in time.
10. Construct a speed-density relationship \( V(\rho) \) with \( dV(\rho)/d\rho \leq 0 \), for which the fundamental diagram has several maxima.

11. Show that \( \rho(x, t) = \rho(x_0, t - (x - x_0)/V_0) \) and \( \rho(x, t) = \rho(x_1, t - (x_1 - x)/c_0) \) with \( x_0 \leq x \leq x_1 \) are solutions of the Lighthill-Whitham-Richards model for free and congested traffic, respectively, if the fundamental diagram is specified in a triangular way as in Eq. (2.53).

12. Start with the traffic model

\[
\frac{\partial}{\partial t}[V + p(\rho)] + V \frac{\partial}{\partial x}[V + p(\rho)] = 0 \tag{2.96}
\]

with \( p(\rho) = \rho^\gamma \) proposed by Aw and Rascle [33], and derive a specification of the traffic pressure \( P(\rho) \) that allows to present this model as a special case of the velocity equation (2.59).

13. What is the microscopic traffic model that corresponds exactly to the Lighthill-Whitham-Richards model?

14. Assume we would establish a micro-macro link between microscopic and macroscopic traffic variables by the definitions

\[
\begin{align*}
\rho(x, t) &= \frac{1}{x_i(t) - x_{i+1}(t)} \left[ x_{i-1}(t) - x + \frac{1}{x_{i-1}(t) - x_i(t)} x - x_i(t) \right], \\
V(x, t) &= \frac{v_i(t) \left[ x_{i-1}(t) - x \right] + v_{i-1}(t) \left[ x - x_i(t) \right]}{x_{i-1}(t) - x_i(t)}, \\
A(x, t) &= \frac{f_i(t) \left[ x_{i-1}(t) - x \right] + f_{i-1}(t) \left[ x - x_i(t) \right]}{x_{i-1}(t) - x_i(t)},
\end{align*}
\tag{2.97}
\]

i.e. assume that the macroscopic variables in the vehicle locations \( x = x_i(t) \) would be given by the microscopic ones, while in locations \( x \) between two vehicles, they would be defined by linear interpolation. Then, derive by partial differentiation of Eq. (2.97) with respect to \( x \) the relationship

\[
\frac{\partial V(x, t)}{\partial x} = \frac{v_{i-1}(t) - v_i(t)}{x_{i-1}(t) - x_i(t)} \tag{2.98}
\]

with \( x_i(t) \leq x \leq x_{i-1}(t) \). Moreover, using this, what is the right-hand side of the velocity equation \((\partial/\partial t + V \partial/\partial x)V = ?\) resulting from the above definitions? Finally, do the above definitions fulfil the continuity equation \( \partial \rho/\partial t + V \partial \rho/\partial x = -\rho \partial V/\partial x \)?

15. Derive the macroscopic velocity equation (2.75) directly from the simple density, velocity, and variance formulas
\[ \rho(x, t) = \sum_i s(x_i(t) - x) \]
\[ V(x, t) = \frac{\sum_i v_i(t)s(x_i(t) - x)}{\rho(x, t)} , \]
\[ \theta(x, t) = \sum_i [\delta v_i(t)]^2 s(x_i(t) - x) \]
\[ \frac{\rho(x, t)}{\rho(x, t)} , \] (2.99)

see Eqs. (2.41), (2.42), and (2.71), assuming a symmetrical smoothing function \( s(x_i - x) = s(x - x_i) = s(|x_i - x|) \). How must \( f(d) \) be chosen to be consistent with the optimal velocity model? What happens, if \( 1/\rho \) in Eq. (??) is chosen large (say, 100 m) and constant? How does the formula (??) for the local density look like? Does it give reasonable values?

16. Replace the zeroth order approximation in our derivation in the section “Micro-Macro-Link” \( f(1/\rho(x + 1/\rho, t)) \approx f(1/\rho(x, t)) \) by a first (or second) order approximation. How does the resulting macroscopic velocity equation look like, if \( f(1/\rho(x+1/\rho, t)) - f(1/\rho(x, t)) \) is expressed through a density gradient? Does any of the resulting models correspond to one of the phenomenological traffic models proposed in Sec. 2.6?

2.9 Further Reading


3

Waves and Shock Fronts

After having introduced some fundamental models that will be discussed in this book, we will now, step by step, analyze their various properties. Our particular interest is dedicated to the spatio-temporal behavior of the model solutions. While some of these behaviors are only accessible by computer simulation, an analytical understanding can be gained to a surprising extent. In this chapter, we will start with analyzing solutions of the Lighthill-Whitham-Richards model. The analysis of models with diffusion and dissipation is the focus of the next chapter, and the impact of driving terms and delays will be the subject of Chap. 5.

3.1 Sound Waves

We will start with an analysis of the dynamical solutions of the continuity equation

$$\frac{\partial \rho(x,t)}{\partial t} + V(x,t) \frac{\partial \rho(x,t)}{\partial x} = -\rho(x,t) \frac{\partial V(x,t)}{\partial x} \quad (3.1) \{g1\}$$

in combination with the one-dimensional velocity equation

$$\frac{\partial V(x,t)}{\partial t} + V(x,t) \frac{\partial V(x,t)}{\partial x} = -\frac{1}{\rho} \frac{\partial P(\rho)}{\partial x} \quad (3.2) \{g2\}$$

These so-called Euler equations [1] can be considered to model frictionless fluid or gas flows in one dimension. Compared to the velocity equation (2.59), we have dropped the driving term \([V_e(\rho) - V]/\tau\), which is typical for traffic flows. Therefore, we do not have an equilibrium velocity-density relation \(V_e(\rho)\), now.

In order to determine the solution of the above equations, we will derive linearized equations for the case of small deviations

$$\delta \rho(x,t) = \rho(x,t) - \rho_e \quad \text{and} \quad \delta V(x,t) = V(x,t) - V_e \quad (3.3) \{lin\}$$

from the stationary and homogeneous solution \(\rho(x,t) = \rho_e\) and \(V(x,t) = V_e = 0\). The quantity \(\rho_e\) corresponds to the average density of the fluid or gas.
Inserting (3.3) into Eqs. (3.1) and (3.2) and neglecting non-linear terms in the small deviations $\delta \rho, \delta V$ results in
\[
\frac{\partial \delta \rho(x,t)}{\partial t} + V_e \frac{\partial \delta \rho(x,t)}{\partial x} = -\rho_e \frac{\partial \delta V(x,t)}{\partial x} \tag{3.4}
\]
and
\[
\frac{\partial \delta V(x,t)}{\partial t} + V_e \frac{\partial \delta V(x,t)}{\partial x} = -\frac{1}{\rho_e} \frac{dP(\rho_e)}{d\rho} \frac{\partial \delta \rho(x,t)}{\partial x} \tag{3.5}
\]
Considering $V_e = 0$ and deriving Eq. (3.4) with respect to $t$, but Eq. (3.5) with respect to $x$ yields
\[
\frac{\partial^2 \delta \rho(x,t)}{\partial t^2} + \rho_e \frac{\partial^2 \delta V(x,t)}{\partial t \partial x} = 0 \tag{3.6}
\]
and
\[
\frac{\partial^2 \delta V(x,t)}{\partial x \partial t} = -\frac{1}{\rho_e} \frac{dP(\rho_e)}{d\rho} \frac{\partial^2 \delta \rho(x,t)}{\partial x^2} \tag{3.7}
\]
Inserting Eq. (3.7) into Eq. (3.6) finally gives the so-called wave equation
\[
\frac{\partial^2 \delta \rho(x,t)}{\partial t^2} - \hat{c}^2 \frac{\partial^2 \delta \rho(x,t)}{\partial x^2} = 0 \tag{3.8}
\]
which is well-known from one-dimensional sound propagation. The constant \(\hat{c} = \sqrt{\frac{dP(\rho_e)}{d\rho}}\) corresponds to the speed of sound.

In order to determine the spatio-temporal solution of Eq. (3.8), we rewrite this equation, inspired by the relationship \((a^2 - b^2) = (a + b)(a - b)\):
\[
\left(\frac{\partial}{\partial t} - \hat{c} \frac{\partial}{\partial x}\right) \left(\frac{\partial}{\partial t} + \hat{c} \frac{\partial}{\partial x}\right) \delta \rho(x,t) = 0 . \tag{3.10}
\]
This is a good trick to make the solution directly visible. The general solution of this wave equation is
\[
\delta \rho(x,t) = F(x - \hat{c}t) + G(x + \hat{c}t) . \tag{3.11}
\]
For example, a superposition of sine waves,
\[
\delta \rho(x,t) = A' \left[ \sin(x - \hat{c}t) + \sin(x + \hat{c}t) \right] , \tag{3.12}
\]
solves Eq. (3.8). This can be easily checked by differentiation and insertion into the wave equation. Hence, its solution is a superposition of a forward and a backward moving wave with velocities $\pm \hat{c}$. Note that the amplitudes of these waves neither grow nor shrink. The concrete form of the solution is determined by the initial conditions $\delta \rho(x,0)$ and $\partial \delta \rho(x,0)/\partial t$:
\[
\delta \rho(x,t) = \frac{\delta \rho(x - \hat{c}t,0) + \delta \rho(x + \hat{c}t,0)}{2} + \frac{1}{2\hat{c}} \int_{x-\hat{c}t}^{x+\hat{c}t} dx' \frac{d\delta \rho(x',0)}{dt} . \tag{3.13}
\]
3.2 Kinematic Waves

Let us now come back to the Lighthill-Whitham-Richards model. According to Eq. (2.52), it can be written
\[
\left( \frac{\partial}{\partial t} + c(\rho) \frac{\partial}{\partial x} \right) \rho(x, t) = 0 \tag{3.14}
\]
with
\[
c(\rho) = \frac{dQ(\rho)}{d\rho} = V_c(\rho) + \rho \frac{dV_c(\rho)}{d\rho} \leq V_c(\rho). \tag{3.15}
\]
Comparing this with Eq. (3.10), it becomes obvious that the LWR equation (3.14) may be viewed as “half” of the wave equation discussed in the previous section. Therefore, rather than two oppositely moving waves we have now a single “kinematic” wave moving with velocity \(c(\rho)\). The general solution is
\[
\rho(x, t) = \rho\left(x - \int_0^t dt' c\left(\rho(y_x(t'), t')\right), 0\right), \tag{3.16}
\]
where
\[
y_x(t') = x - \int_{t'}^{t} dt'' v(t'') \quad \text{with} \quad \frac{dy_x(t')}{dt'} = v(t') = c\left(\rho(y_x(t'), t')\right) \tag{3.17}
\]
is the location with vehicle density \(\rho\) as a function of time \(t'\). As this defines an implicit equation for the density \(\rho(x, t)\), it is hard to solve explicitly (if \(c\) is not a constant). However, we can see from Eq. (3.16) that the initial density profile \(\rho(x, 0)\) does not change its amplitude in the course of time.\(^1\)

In case of a density-independent propagation speed \(c\), equations simplify considerably. Then, a smooth wave profile propagates at constant speed without changing its shape. In Sec. 2.5 we have already discussed this for a triangular fundamental diagram with the slopes \(V_0\) and \(c_0\), which correspond to the characteristic speeds. Then, we have the simple solution
\[
\rho(x, t) = \rho(x_0 - V_0 t, 0) \tag{3.18}
\]
for \(x > x_0\) in the free flow regime with \(\rho(x, t) < \rho_c\), while
\[
\rho(x, t) = \rho(x_1 - c_0 t, 0) = \rho(x_1 + |c_0| t, 0) \tag{3.19}
\]
holds for \(x < x_1\) in the congested flow regime. That is, the wave moves either upstream or downstream but not both, in contrast to sound waves.

3.3 Shock Waves

We have seen that the LWR model, like the sound wave model, implies constant wave amplitudes during their one-dimensional motion. However, the

\(^1\) According to [2], however, the interaction of shockwaves can lead to a reduction of the amplitude. The opposite, i.e., an increase of the amplitude has, to our knowledge, has nowhere been reported.
shape of kinematic waves changes, if the function \( c(\rho) \) is not constant. Let us assume that \( c(\rho) \) becomes smaller with increasing density (see Fig. 3.1). Then, areas of low density move faster than areas of high density. Consequently, if the density has a positive gradient \( \partial \rho(x,t)/\partial x > 0 \) along the road, the gradient will further increase. Eventually, a shock wave evolves (see Fig. 3.2), which leads to a jump in the density, average velocity, and flow \([3, 4]\). While the speed \( c(\rho) \) of the kinematic waves was determined by the slope \( dQ_e(\rho)/d\rho \) of the tangent to the fundamental diagram (see Eq. (3.15) and Fig. 3.3), the speed \( C(\rho_-, \rho_+) \) of shock waves is given by the secant:

\[
C(\rho_-, \rho_+) = \frac{Q_e(\rho_+) - Q_e(\rho_-)}{\rho_+ - \rho_-}.
\] (3.20)

Herein, \( \rho_+ \) represents the density immediately downstream of the shock wave, while \( \rho_- \) is the density immediately upstream. Formula (3.20) follows again from the conservation of the number of vehicles.

Note that the velocities of the vehicles, the kinematic waves, and the shock waves can be graphically constructed from the flow-density relationship (see Fig. 3.3). This is certainly one of the reasons for calling this relationship the “fundamental diagram”.

**Fig. 3.1.** Illustration of the density-dependence of the propagation velocity \( c(\rho) \) of kinematic waves, as derived from the fundamental diagram \( Q_e(\rho) \).
3.3 Shock Waves

Fig. 3.2. Illustration of the change of the density profile and the development of a shock wave according to the Lighthill-Whitham-Richards model.

Fig. 3.3. Illustration of the fundamental diagram $Q_e(\rho)$ and of the quantities that can be graphically derived from it, namely the average vehicle speed $V_e(\rho)$, the kinematic wave velocity $c(\rho) = dQ_e(\rho)/d\rho$, and the speed of shock waves, $C(\rho_-,\rho_+) = [Q_e(\rho_+) - Q_e(\rho_-)]/(\rho_+ - \rho_-)$. 
3.4 Emergent Waves

By now, the reader is certainly curious, under what conditions it is possible to have growing wave amplitudes as well. Indeed, the additional driving term in the macroscopic traffic models can lead to emergent waves, i.e. under certain conditions, small perturbations (i.e. variations) in the traffic flow can grow and finally generate a traffic jam. These jams have sometimes been called “phantom traffic jams”, as they seem to appear without any significant reason such as a lane closure, an accident, or a construction site. The instability mechanism which can trigger “phantom traffic jams” is discussed in Sec. 5.1.

I should, however, mention already that “phantom traffic jams” are more a theoretical possibility than a description of what really causes traffic jams on freeways. This is because, in open street systems (rather than on circular roads with periodic boundary conditions), traffic usually does not enter the regime of linear instability required for the occurrence of “phantom traffic jams”. According to Treiber [5], real traffic jams are caused by a combination of three “ingredients” (see Sec. 15.1):

1. a high traffic volume,
2. a bottleneck (e.g. an on-ramp, a reduction in the number of lanes, or a gradient), and
3. a perturbation in the traffic flow.

We will come back to this in Sec. ??.

3.5 Exercises

1. Derive the linearized set of partial differential equations (3.4), (3.5) from the non-linear partial differential equations (3.1) and (3.2) step by step, i.e. insert the formulas for the small density and velocity deviations \( \delta \rho(x, t) \) and \( \delta V(x, t) \), carry out Taylor expansions of first order, show the disappearance of zeroth order terms, and drop second and higher order terms.

2. Show by differentiation that (3.11) solves the wave equation (3.8). Moreover, check that formula (3.19) constitutes a special solution of the wave equation (3.8) which is compatible with the initial conditions \( \delta \rho(x, 0) \) and \( \partial \delta \rho(x, 0) / \partial t \). Why do we need two initial conditions (for the density and its derivative) to have a unique specification? Determine the special solution in the case (a) \( \delta \rho(x, 0) = \exp(-x^2) \), \( \partial \delta \rho(x, 0) / \partial t = 0 \) and (b) \( \delta \rho(x, t) = 0 \), \( \partial \delta \rho(x, 0) / \partial t = \cos(x) \).

3. Differentiate Eq. (3.16) partially with respect to \( x \) and \( t \), considering Eq. (3.17), and insert the resulting expressions into Eq. (3.14) in order to show that Eqs. (3.16), (3.17) are a special solution of the Lighthill-Whitham-Richards model. Show that this solution is also consistent with the initial condition \( \rho(x, 0) \). Simplify the solution of the LWR model for the case \( c(\rho) = c = \text{const} \). Check that Eqs. (3.18) and (3.19) result in the case of the triangular fundamental diagram (2.53).
4. Derive the speed of shock waves from the conservation of the number of vehicles, assuming the densities $\rho_-$ and $\rho_+$ directly downstream and upstream of the shock front.

3.6 Further Reading

Before we enter the subject of unstable flow dynamics, we should first understand the impact of dissipation effects. For this, let us start with the linear diffusion equation.

### 4.1 The Diffusion Equation

In one spatial dimension, the diffusion equation is a partial differential equation of the form

$$\frac{\partial \rho(x,t)}{\partial t} = D \frac{\partial^2 \rho(x,t)}{\partial x^2},$$

(4.1)  \{diffus\}

where $D > 0$ is the diffusion coefficient \[1\]. For example, if $\rho(x,t)$ represents the density of ink particles in a resting, one-dimensional column of fluid, the above equation describes the spreading of ink particles along the column based on random molecular collisions. Naturally, the rate of random motion out of denser areas into areas of low ink concentration is higher than the (opposite) flow against the density gradient. Therefore, the system has a tendency to evolve towards an uniform spatial distribution of ink.

It turns out that diffusion processes play an important role in physics, chemistry, and biology \[2\]. Having a closer look at the diffusion equation (4.1), we can immediately see that the density $\rho(x,t)$ grows in time, where its second spatial derivative $\partial^2 \rho(x,t)/\partial x^2$ is positive. This applies to areas in which the spatial gradient of the density is growing, i.e. particularly to minima of the density. The density is reduced in time where the second spatial derivative is negative, i.e. in areas where the spatial gradient of the density decreases. This particularly applies to maxima of the density. For these reasons, the diffusion term has the initially mentioned smoothing effect on the density. (The opposite would be true for a negative diffusion coefficient. In that case, the diffusion term would have a destabilizing effect, the density would be strongly fluctuating in space after a short time period.)

It can be easily shown by differentiation that
\[ \rho(x, t) = \frac{1}{\sqrt{4\pi D t}} e^{-\frac{(x-x')^2}{4Dt}} \] (4.2)

is a special solution of the diffusion equation (4.1) for each value of \( x' \). Due to the linearity of the diffusion equation, a superposition of such functions is a solution as well. If a certain initial condition \( \rho(x, 0) \) is given, the solution reads

\[ \rho(x, t) = \frac{1}{\sqrt{4\pi D t}} \int_{-\infty}^{\infty} dx' \rho(x', 0) \exp \left( -\frac{(x-x')^2}{4Dt} \right) . \] (4.3)

That is, the initial concentration \( \rho(x, 0) \) at each place \( x \) diffuses independently according to a normal distribution. By integration one can show that the variance is given by

\[ \int_{-\infty}^{\infty} dx (x-x_0)^2 \rho(x, t) = 2Dt , \] (4.4)

if all elements are initially concentrated in one location \( x = x_0 \), because in this case, we simply have \( \rho(x, 0) = \delta(x-x_0) \) and

\[ \rho(x, t) = \frac{1}{\sqrt{4\pi D t}} e^{-\frac{(x-x_0)^2}{4Dt}} . \] (4.5)

The linear growth of the variance in time is typical for random processes characterized by a Gaussian white noise [3, 4]. However, some random processes like, for example, “Levy flights” can lead to anomalous diffusion [5]. This is characterized by a variance that grows in a power-law manner, i.e. proportionally to \( t^\alpha \). The case \( \alpha > 1 \) is referred to as superdiffusion, while cases with \( \alpha < 1 \) are known as subdiffusion. Diffusive processes of this kind can be described by fractional differential equations, which contain fractional derivatives [6, 7]. These may reflect, for example, the fractal nature of the substrate on which particle transport takes place (e.g. porous media like coffee) or the special distribution of the time intervals between subsequent jumps between different locations [8, 9]. In particular, there are interesting applications to financial problems (such as prices of stocks) [10, 11] and to human travel behavior [12].

### 4.2 The Burgers Equation

Let us now return to the partial differential equation

\[ \frac{\partial \rho(x, t)}{\partial t} + \frac{\partial Q(x, t)}{\partial x} = D \frac{\partial^2 \rho(x, t)}{\partial x^2} , \] (4.6)

which we already encountered in Eq. (2.66). This equation is a Lighthill-Whitham-Richard model with an additional diffusion term \( D \partial^2 \rho(x, t)/\partial x^2 \).
Compared to the diffusion equation (4.1), it contains the additional term \( \partial / \partial x [\rho(x,t) V_c(\rho(x,t))] \), which makes the above partial differential equation non-linear. In some sense, it is the simplest partial differential equation which contains both, a non-linear transport term and diffusion.

\[ V_c(\rho) = V_0 \left( 1 - \frac{\rho}{\rho_{\text{max}}} \right). \] (4.7) \{Greens\}

The relationship \( c(\rho) = \partial Q_c(\rho) / \partial \rho \) for the propagation speed leads to

\[ c(x,t) = V_0 \left( 1 - \frac{2\rho(x,t)}{\rho_{\text{max}}} \right). \] (4.8)

With this, equation (4.6) can be transformed into the simpler Burgers equation

\[ \frac{\partial c(x,t)}{\partial t} + c(x,t) \frac{\partial c(x,t)}{\partial x} = D \frac{\partial^2 c(x,t)}{\partial x^2}. \] (4.9) \{burg\}

Furthermore, by means of the Cole-Hopf transformation

\[ c(x,t) = -\frac{2D}{\Psi(x,t)} \frac{\partial \Psi(x,t)}{\partial x}, \] (4.10) \{Hopf\}
Eq. (4.9) becomes just a linear diffusion equation:
\[
\frac{\partial \Psi(x,t)}{\partial t} = D \frac{\partial^2 \Psi(x,t)}{\partial x^2} \quad (4.11)
\]

In order to show this, we follow the presentation by Whitham [13]: First we define a function \( \Psi'(x,t) \) by
\[
\frac{\partial \Psi'(x,t)}{\partial x} := c(x,t). \quad (4.12)
\]

If we insert this into the Burgers equation (4.9) and consider
\[
\frac{1}{2} \frac{\partial}{\partial x} \left( \frac{\partial \Psi'}{\partial x} \right)^2 = \frac{\partial \Psi'}{\partial x} \frac{\partial^2 \Psi'}{\partial x^2}, \quad (4.13)
\]
by integration over \( x \) we obtain
\[
\frac{\partial \Psi'}{\partial t} + \frac{1}{2} \left( \frac{\partial \Psi'}{\partial x} \right)^2 = D \frac{\partial^2 \Psi'}{\partial x^2}. \quad (4.14)
\]

Finally, by means of the transformation
\[
\Psi(x,t) = -\frac{\Psi'(x,t)}{2D}, \quad (4.15)
\]
we find the linear diffusion equation. As in the previous section, its general solution is
\[
\Psi(x,t) = \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} dx' \Psi(x',0) \exp\left(-\frac{(x-x')^2}{4Dt}\right). \quad (4.16)
\]

From this, the solution of the Burgers equation results with the transformation (4.10) and the initial condition
\[
\Psi(x,0) = \Psi(0,0) \exp\left(-\frac{1}{2D} \int_{0}^{x} dx' c(x',0)\right). \quad (4.17)
\]

Compared to the Lighthill-Whitham-Richards equation, the Burgers equation describes smoothed shock fronts due to the additional diffusion term. This avoids discontinuities in the density profile, which make the numerical solution of the LWR equation non-trivial [14]. In the limit of vanishing diffusion, there are only a few numerical integration methods that cope well with the then occurring shock waves. The Godunov scheme [15] is probably the most well-known of these.
4.3 Exercises

1. Please check by differentiation that Eq. (4.3) is a solution of the diffusion equation (4.1) and consistent with the initial condition $\rho(x,0)$. Assume $\rho(x,0) = \delta(x-x_0)$ and derive the variance for this case step by step. Is it possible to find a formula for the time-dependent variance for the case of arbitrary initial conditions $\rho(x,0)$ as well?

2. The following generalized diffusion equation has been proposed to describe the dynamics of innovations [16]:

$$\frac{\partial \rho(x,t)}{\partial t} = \lambda \rho(x,t) \left[ x - \langle x \rangle_t \right] + D \frac{\partial^2 \rho(x,t)}{\partial x^2}. \quad (4.18) \{\text{innov}\}$$

In this equation, $x$ represents the “fitness”, i.e. the success of a technology, $\rho(x,t) \, dx$ the probability of finding a fitness value between $x$ and $x + dx$, and

$$\langle x \rangle_t = \int_0^\infty dx \, x \rho(x,t) \quad (4.19)$$

is the average fitness. The term in square brackets describes an imitation of above-average technologies and the eventual disappearance of below-average strategies. The parameter $\lambda$ reflects something like the “selection pressure”, while the diffusion term delineates the effect of random mutations (good or bad “innovations”). Please show by differentiation with respect to $x$ and $t$ that

$$\rho(x,t) = \frac{1}{\sqrt{2\pi g(t)}} e^{-[x-f(t)]^2/[2g(t)]} \quad (4.20)$$

is a special solution of Eq. (4.18), if $f(t)$ and $g(t)$ satisfy certain differential equations. How do these differential equations look like, if $D = \text{const}$? What is the solution of these differential equations? How would the solution look like, if $D(t) = C_0 g(t)$ or $D(t) = C_1 f(t)$ with positive constants $C_0$ and $C_1$?

3. Reproduce yourself all the single steps in the derivation of the solution of the Burgers equation for a linear velocity-density relationship.

4.4 Further Reading

Driving Term, Delays and Instability

While the Lighthill-Whitham-Richard and the Burgers equation can describe queuing effects of vehicles, they are not capable of reproducing more complex congestion patterns, as we observe them on freeways (see Fig. ?? and Sec. ...). Therefore, an important element of traffic dynamics must be missing in the previous models. Consequently, we will now study the delayed adaptation to changes in the traffic situation, which is reflected by the driving term \[ V_c(\rho) - V \]/\tau. \footnote{Note that, in some sense, the stochastic instability mechanism underlaying the cellular automaton model of Nagel and Schreckenberg (see Sec. ??) is based on delayed adaptation as well: With a certain probability, it assumes a reduction in the speed compared to the safe speed, which is given by the vehicle gap. Therefore, it takes an additional time step to catch up with the leading vehicle again.}

We will now analyze the stability of microscopic and macroscopic traffic models with delayed adaptation. In particular, we will derive the specific instability conditions, under which emergent waves are expected to occur. If these instability conditions are fulfilled, even small perturbations will eventually grow and give rise to spatio-temporal patterns. The formation of these patterns are due to the combination of the driving term, which generates a continuous inflow of energy into the system, with inhomogeneous dissipation effects that may depend on local interactions. In this way, we have a steady re-distribution of energy in the system, which leads to spatio-temporal self-organization phenomena (see Sec. ??).

The driving term alone without any interaction effects (i.e. in the case \( V_c(\rho) = V^0 \)) would not lead to any interesting dynamics. Obviously, the acceleration equation\footnote{For simplicity, we use the representation with a total time derivative \( d/dt = \partial/\partial t + V \partial/\partial x \) here, which corresponds to the dynamics from the driver’s perspective.}

\[
\frac{dV(x, t)}{dt} = \frac{V^0 - V(x, t)}{\tau} \tag{5.1} \]

would only imply an exponential adjustment to the desired speed \( V^0 \):
Driving Term, Delays and Instability

\[ V(x, t) = V^0 + [V(x, 0) - V^0]e^{-t/\tau} \]  

(5.2)

(see Exercise ?? of Chapter ??). Accordingly, the system would eventually approach the constant solution \( V(x, t) = V^0 \), i.e. the system behavior would be stationary and homogeneous. It is hard to imagine a more boring case! Therefore, it is surprising that the combination of the driving term (5.1) describing exponential decay with the Euler equation (??) describing sound waves of constant amplitude potentially leads to waves of growing amplitude, as we shall show in the next sections. Obviously, there must be interaction effects between different terms of the model which re-distribute the energy pumped into the system via the driving term.

5.1 Linear Instability in Macroscopic Traffic Models

5.1.1 Linearization of the Traffic Equations

In order to understand the dynamics of traffic flows, it is important to find out whether and under what conditions variations in the traffic flow can grow and eventually cause traffic congestion. In fact, we will see that delay-induced instabilities can lead to phenomena such as “phantom traffic jams” (see Figs. 5.1 and 5.2) or “business cycles” (see Sec. ??), depending on the system we study. Let us start with the discussion of the first case, based on the macroscopic, Navier-Stokes-like traffic model given by the continuity equation

\[ \frac{\partial \rho(x, t)}{\partial t} + V(x, t) \frac{\partial \rho(x, t)}{\partial x} = -\rho(x, t) \frac{\partial V(x, t)}{\partial x} \]  

(5.3)

and the velocity equation

\[ \frac{\partial V(x, t)}{\partial t} + V(x, t) \frac{\partial V(x, t)}{\partial x} = -\frac{\theta_0}{\rho(x, t)} \frac{\partial \rho(x, t)}{\partial x} + \frac{\eta}{\rho(x, t)} \frac{\partial^2 V(x, t)}{\partial x^2} + \frac{1}{\tau} [V_\rho(\rho) - V(x, t)] , \]  

(5.4)

see Eq. (2.68). Our stability analysis is carried out for a circular road of length \( L \), as this allows us to apply periodic boundary conditions. First, we must determine the stationary and homogeneous solution of Eqs. (5.3) and (5.4), which are defined by the vanishing temporal and spatial derivatives, respectively. While stationarity assumes \( \partial \rho/\partial t = 0 \) and \( \partial V/\partial t = 0 \), homogeneity (i.e. independence of space) assumes \( \partial \rho/\partial x = 0 \), \( \partial V/\partial x = 0 \), and \( \partial^2 V/\partial x^2 = 0 \).

The stationary and homogeneous solution is given by

\[ \rho(x, t) = \rho_e, \quad V(x, t) = V_e(\rho_e) , \]  

(5.5)

where
5.1 Linear Instability in Macroscopic Traffic Models

Fig. 5.1. Simulation of the Kerner-Konhäuser model on a circular road of length $L = 10$ km with the assumed speed-density relationship $V_e(\rho) = V^0\left\{\left[1 + \exp\left((\rho/\rho_{\text{max}} - 0.25)/0.06\right)\right]^{-1} - 3.72 \cdot 10^{-6}\right\}$, where $V^0 = 144$ km/h, $\rho_{\text{max}} = 168$ veh./km, $\tau = 11$ s, $\sqrt{\theta^0} = 54$ km/h, $\eta_0 = 436$ km/h. Although the initial traffic situation is almost homogeneous (apart from a small sinusoidal perturbation of wavelength $L$), we eventually find emergent traffic jams with a distance smaller than $L$, which propagate upstream, i.e. with negative velocity.

$$\rho_e = \frac{1}{L} \int_0^L dx \, \rho(x, t) = \frac{N}{L} \quad (5.6)$$

is the average vehicle density, which is constant in time on the assumed circular road. We want to find out whether deviations

$$\delta \rho(x, t) = \rho(x, t) - \rho_e \quad \text{and} \quad \delta V(x, t) = V(x, t) - V_e(\rho_e) \quad (5.7)$$
Fig. 5.2. Trajectories of virtual cars in the simulation scenario of Fig. 5.1, determined via the formula \( x_i(t + dt) = x_i(t) + V(x_i(t), t) \, dt \), where \( dt \) represents a small time step. The emergent traffic jams are characterized by a larger density of the trajectories and a drop of their slope (velocity) to values of approximately zero. Note that traffic jams propagate backwards, i.e. opposite to the direction of vehicle motion.

grow or decay in time. When studying this, we will restrict ourselves to “small” deviations defined by

\[
\delta \rho(x, t) \ll \rho_e, \quad \text{and} \quad \delta V(x, t) \ll V_e(\rho_e).
\]  

Under such conditions, we can restrict our analysis to linearized partial differential equations, as quadratic and higher order terms are at least one order smaller than the linear ones. We prefer to work with linear equations, as there exists a fully elaborated theory of how to solve the equations. For example, the superposition (sum) of two special solutions of linear equations is again a solution of these equations. Moreover, calculations are much simpler than for non-linear equations (see Sec. ??).

To obtain the linearized partial differential equations, we insert \( \rho(x, t) = \rho_e + \delta \rho(x, t) \) and \( V(x, t) = V_e(\rho_e) + \delta V(x, t) \) into the macroscopic equations (5.3) and (5.4), and neglect terms that are not linear in \( \delta \rho(x, t) \) or \( \delta V(x, t) \). Since the terms of zeroth order vanish in the stationary and homogeneous state, we finally remain with the following linearized set of partial differential equations:

\[
\frac{\partial \delta \rho}{\partial t} + V_e(\rho_e) \frac{\partial \delta \rho}{\partial x} = -\rho_e \frac{\partial \delta V}{\partial x}.
\]  

\[(\text{linrho})\]

and

\[
\frac{\partial \delta V}{\partial t} + V_e(\rho_e) \frac{\partial \delta V}{\partial x} = -\theta_0 \frac{\partial \delta \rho}{\partial x} + \frac{\eta}{\rho_e} \frac{\partial^2 \delta V}{\partial x^2}
+ \frac{1}{\tau} \left[ \frac{dV_e(\rho_e)}{d\rho} \delta \rho(x, t) - \delta V(x, t) \right].
\]  

\[(5.10)\]
5.1.2 Solution Ansatz and Complex Numbers

It is useful to make the solution ansatz

\[
\delta \rho(x, t) = \delta \rho_0 \exp \left( i \kappa x + (\lambda - i \omega) t \right) = \delta \rho_0 e^{\lambda t} e^{i(\kappa x - \omega t)},
\]

\[
\delta V(x, t) = \delta V_0 \exp \left( i \kappa x + (\lambda - i \omega) t \right) = \delta V_0 e^{\lambda t} e^{i(\kappa x - \omega t)}. \quad (5.11)
\]

\[\text{Fig. 5.3. Illustration of a complex number } z = x + iy \text{ with } x = r \cos(\phi) \text{ and } y = r \sin(\phi) \text{ in the complex plane, where } r = \sqrt{x^2 + y^2} \text{ and } \tan(\phi) = y/x. \text{ } i \text{ denotes the complex unit. TO BE ADDED, WENJIAN}\]

In order to understand these formulas well, let us shortly repeat some definitions and properties of complex numbers. A complex number

\[z = x + iy = r e^{i \phi} = r \cos(\phi) + i r \sin(\phi) \quad (5.12)\]

can be represented in two-dimensional space with coordinates \(x = \text{Re}(z) = r \cos(\phi)\) and \(y = \text{Im}(z) = r \sin(\phi)\), which are called the real part and the imaginary part of the complex number (see Fig. 5.3). The absolute value is given as

\[r = \sqrt{x^2 + y^2} = \sqrt{(x + iy)(x - iy)} = \sqrt{z \overline{z}}, \quad (5.13)\]

where \(\overline{z} = x - iy = re^{-i\phi}\) is called the conjugate complex number. The angle \(\phi\) is determined by

\[\tan(\phi) = \frac{\sin(\phi)}{\cos(\phi)} = \frac{y}{x} = \frac{\text{Im}(z)}{\text{Re}(z)}, \quad (5.14)\]

and the exponential functions is defined as for real numbers by the infinite series expansion

\[\exp(z) = e^z = \sum_{l=0}^{\infty} \frac{z^l}{l!}, \quad (5.15)\]

where \(l! = l \cdot (l - 1) \ldots 2 \cdot 1\). Therefore, the relationships for exponential functions apply also to the case of complex numbers, i.e. the product of two complex numbers \(z_1 = x_1 + iy_1 = r_1 e^{i\varphi_1}\) and \(z_2 = x_2 + iy_2 = r_2 e^{i\varphi_2}\) is given by

\[z_1 z_2 = (x_1 y_1 - y_1 y_2) + i(x_1 y_2 + y_1 x_2) = r_1 e^{i\varphi_1} r_2 e^{i\varphi_2}
= r_1 r_2 e^{i(\varphi_1 + \varphi_2)} = r_1 r_2 \cos(\varphi_1 + \varphi_2) + ir_1 r_2 \sin(\varphi_1 + \varphi_2). \quad (5.16)\]

As the real and imaginary part are linearly independent of each other, this implies \(x_1 y_1 - y_1 y_2 = r_1 r_2 \cos(\varphi_1 + \varphi_2)\) and \(x_1 y_2 + y_1 x_2 = r_1 r_2 \sin(\varphi_1 + \varphi_2)\).

The inverse of a complex number is given by
The imaginary unit $i$ has the property $i^2 = -1$ and may, therefore, be written as $i = \sqrt{-1} = e^{i\pi/2}$.

Because of $\exp(i\kappa x) = \cos(\kappa x) + i\sin(\kappa x)$, the ansatz (5.11) assumes that the perturbation of the stationary and homogeneous traffic situation can be represented as a periodic function with the wave number $\kappa$ and wavelength $2\pi/\kappa$. The wave frequency of Eq. (5.11) is $\omega$, while $\rho_0 \exp(\lambda t)$ and $V_0 \exp(\lambda t)$ are the amplitudes at time $t$. That is, if the “growth rate” $\lambda$ is greater than zero, even small perturbations will eventually grow, which can give rise to “phantom traffic jams” (see Figs. 5.1 and 5.2). For $\lambda < 0$, however, the initial perturbation will be damped out and the stationary and homogeneous solutions will be re-established, i.e. it is stable with respect to small perturbations.

Below we will see that, for each specification of $\kappa$ and the average density $\rho_e$, there exist two solutions $l \in \{+, -\}$ with the frequencies $\omega_l(\kappa)$ and the growth rates $\lambda_l(\kappa)$. All the corresponding specifications of ansatz (5.11) are solutions of the linearized partial differential equations. The same applies to their superpositions. The general solution for an arbitrary initial perturbation is of the form

$$
\delta \rho(x, t) = \sum_{l \in \{+, -\}} \int d\kappa \, \delta \rho_0^l(\kappa) \exp \left( i\kappa x + [\lambda_l(\kappa) - i\omega_l(\kappa)]t \right),
$$

(5.18)

$$
\delta V(x, t) = \sum_{l \in \{+, -\}} \int d\kappa \, \delta V_0^l(\kappa) \exp \left( i\kappa x + [\lambda_l(\kappa) - i\omega_l(\kappa)]t \right).
$$

(5.19)

Both, the imaginary and real parts of these expressions are real solutions of the linearized partial differential equation and, therefore, their superposition is a real-valued solution as well. The representation (5.18) may, by the way, be considered as Fourier expansion of the general solution of the linearized partial differential equations. Considering Eq. (5.12), the interpretation of a Fourier expansion is the decomposition of the solution function into sine and cosine functions of different frequency $\omega$, the amplitudes of which may exponentially grow or decay (if $\lambda_l(\kappa) \neq 0$).

### 5.1.3 Solution of the Related Eigenvalue Problem

In order to find the possible $\kappa$-dependent wave numbers $\omega$ and growth rates $\lambda$, we insert ansatz (5.11) into the linearized macroscopic traffic equations (5.9) and (5.10) and use the relationship $i^2 = -1$. The result can represented as an eigenvalue problem:

$$
\begin{pmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{pmatrix}
\begin{pmatrix}
\delta \rho_0 \\
\delta V_0
\end{pmatrix} = 0.
$$

(5.20)
where
\[
M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} = \begin{pmatrix} -\tilde{\lambda} & -i \kappa \rho_e \\ -\frac{i \kappa \theta_0}{\rho_e} + \frac{1}{\tau} \frac{dV_e(\rho_e)}{d\rho} & -\tilde{\lambda} - \frac{\eta}{\rho_e} \kappa^2 - \frac{1}{\tau} \end{pmatrix}.
\] (5.21)

Here, we have used the abbreviation
\[
\tilde{\lambda} = \lambda - i \tilde{\omega} \quad \text{with} \quad \tilde{\omega} = \omega - \kappa V_e(\rho_e).
\] (5.22)

Equation (5.20) is fulfilled only for certain values of \(\tilde{\lambda}(\kappa)\), the “eigenvalues”. These depend on the average density \(\rho_e\) and solve the characteristic polynomial of second order in \(\tilde{\lambda}\), which is obtained by determining the determinant
\[
\det(M) = M_{11} M_{22} - M_{21} M_{12}
\] (5.23)
of the matrix \(M\) and requiring that it becomes zero. The characteristic polynomial reads
\[
(\tilde{\lambda})^2 + \tilde{\lambda} \left( \frac{\eta}{\rho_e} \kappa^2 + \frac{1}{\tau} \right) + i \kappa \rho_e \left[ -\frac{i \kappa \theta_0}{\rho_e} + \frac{1}{\tau} \frac{dV_e(\rho_e)}{d\rho} \right] = 0
\] (5.24)
and has the two solutions
\[
\tilde{\lambda}(\kappa) = \tilde{\lambda}_\pm(\kappa) = -\frac{1}{2 \tilde{\tau}(\kappa)} \pm \sqrt{R(\kappa) \pm i |\Im(\kappa)|}
\] (5.25)
with
\[
\frac{1}{\tilde{\tau}(\kappa)} = \frac{\eta \kappa^2}{\rho_e} + \frac{1}{\tau} \geq 0,
\]
\[
R(\kappa) = \frac{1}{4 \tilde{\tau}^2} - \kappa^2 \theta_0,
\]
\[
\pm |\Im(\kappa)| = \frac{\kappa \rho_e}{\tau} \left| \frac{dV_e(\rho_e)}{d\rho} \right|,
\] (5.26)
where we have assumed the typical case \(dV_e(\rho)/d\rho \leq 0\).

Unfortunately, the square root contains a complex number, which makes it difficult to see the sign of the real value \(\lambda\) of \(\tilde{\lambda}\). However, we can apply the very useful equation
\[
\sqrt{R \pm i |\Im|} = \frac{1}{2} \left( \sqrt{R^2 + |\Im|^2 + R} \right) \pm i \frac{1}{2} \left( \sqrt{R^2 + |\Im|^2 - R} \right).
\] (5.27)
which can be easily checked. From this and Eq. (5.25), we get the following equation for the real part of the eigenvalues:
\[
\lambda_{\pm}(\kappa) = \Re(\tilde{\lambda}_{\pm}(\kappa)) = -\frac{1}{2 \tilde{\tau}} \pm \sqrt{\frac{1}{2} \left( \sqrt{R^2 + |\Im|^2 + R} \right)}.
\] (5.28)
5.1.4 Derivation of the Instability Condition

A transition from stable to unstable behavior, i.e. the change from negative to positive values of $\lambda_\pm(\kappa)$ occurs only for one of the eigenvalues, namely under the condition

$$\lambda_+(\kappa) = -\frac{1}{2\tau} + \sqrt{\frac{1}{2} \left( \sqrt{\Re^2 + \Im^2} \right)} = 0.$$  (5.29)

This implies

$$\left( \frac{1}{4\tau^2} - \frac{\Re^2}{2} \right)^2 = \frac{1}{4}(\Re^2 + \Im^2)$$  (5.30)

and, therefore,

$$\frac{1}{16\tau^4} = \frac{\Re^2}{4\tau^2} + \frac{\Im^2}{4}.$$  (5.31)

Inserting the above definitions of $\Im$ and $\Re$ finally gives

$$\frac{\kappa^2\theta_0}{4\tau^2} = \frac{1}{4} \frac{\kappa^2\rho_e^2}{\tau^2} \left| \frac{dV_e(\rho_e)}{d\rho} \right|^2.$$  (5.32)

As a consequence, the macroscopic traffic equations are unstable under the condition

$$\rho_e \left| \frac{dV_e}{d\rho} \right| > \sqrt{\theta_0} \left( 1 + \frac{\eta\kappa^2}{\rho_e} \right).$$  (5.33)

We notice that this condition for $\text{Re}(\lambda_+) > 0$ is not fulfilled, if the average velocity $V_e(\rho)$ changes little with the density $\rho$, which is usually the case for small and large densities. However, $\lambda_+(\kappa)$ may be greater than zero, i.e. traffic flow may be linearly unstable at medium densities, where $|dV_e(\rho_e)/d\rho|$ tends to be large (see Fig. 5.4). The instability mechanism is based on a reduction of the average velocity with increasing density. According to the continuity equation, this tends to cause a further compression. However, at certain densities, this tendency is overcompensated for by the effect of “traffic pressure” $\rho\theta_0$ (and by the smoothing effect of the viscosity). Note that a finite viscosity coefficient $\eta > 0$ raises the stability threshold for large wave numbers $\kappa$ (i.e. small wave lengths $2\pi/\kappa$) and, hence, makes traffic flow more stable with respect to perturbations. However, the viscosity does not change the stability threshold for perturbations of large wavelength. For these, the stability threshold corresponds to the value $\sqrt{\theta_0}$, which is found for the case $\eta = 0$ of no viscosity at all. Therefore, as favorable as a viscosity term may be from the perspective of numerical simulation, it is not needed from a theoretical point of view (and actually hard to justify).

Note that the independence of Eq. (5.33) from $\tau$ in case of $\eta = 0$ may be misleading. In fact, a more realistic specification of the pressure would be

$$P(x, t) = \rho(x, t)\theta(x, t) \frac{V^0 - V_e(\rho(x, t))}{2\tau},$$  (5.34)
Fig. 5.4. Illustration of the velocity-density diagram and the medium-density regime which is characterized by linear instability, i.e. the growth of small perturbations of traffic flow. TO BE ADDED.

as Eqs. (2.73) and (15.69) show. Therefore, there is a direct dependence of the instability threshold on the delay $\tau$ in adaptation of the actual velocity $V$ to the equilibrium velocity $V_e(\rho)$, see Eq. (15.203). Generally, large values of $\tau$ tend to imply a larger instability regime.

5.1.5 Derivation of the Group Velocity

Let us now study the propagation speed of small perturbations. These are given by the group velocities

$$c_l(\kappa) = \frac{\partial \omega_l(\kappa)}{\partial \kappa}$$

(see Exercise 5.???). Remember that

$$-i\tilde{\omega}_l(\kappa) = -i[\omega_l(\kappa) - \kappa V_e(\rho_e)] = \pm i \sqrt{\frac{1}{2} \left( \sqrt{\Re^2 + \Im^2} - R \right)},$$

see Eqs. (??), (??), and (??). According to Eq. (5.26), in the limit $\tau \to \infty$ we have

$$\frac{1}{\hat{\tau}(\kappa)} = \frac{\eta \kappa^2}{\rho_e} \geq 0,$$

$$\Re(\kappa) = \frac{1}{4 \hat{\tau}^2} - \kappa^2 \theta_0,$$

$$\pm |\Im(\kappa)| = 0,$$

(5.37)

Therefore,

$$\sqrt{\frac{1}{2} \left( \sqrt{\Re^2 + \Im^2} - R \right)} = \sqrt{\frac{1}{2} \left( |\Re| - |\Im| \right)} = \begin{cases} -\Re & \text{if } \Re < 0, \\ 0 & \text{otherwise}. \end{cases}$$

(5.38)

For $\Re < 0$ this implies

$$c_{\pm}(\kappa) = V_e(\rho_e) \mp \frac{d}{d\kappa} \sqrt{-\Re(\kappa)} = V_e(\rho_e) \mp \sqrt{\theta_0 - \frac{\eta}{4 \rho_e}},$$

otherwise $c_{\pm}(\kappa) = V_e(\rho_e)$. If, furthermore, the viscosity vanishes ($\eta = 0$), we should find a relationship which is compatible with Eq. (3.9), as for $\tau \to \infty$ and $\eta = 0$ we obtain the Euler equations investigated in Sec. (??). Considering

$$c_{\pm}(\kappa) - V_e(\rho_e) = \mp \sqrt{\theta_0} = \mp \frac{dP}{d\rho}$$

(5.40)
[see Eq. (2.67)], in the framework moving with the vehicles at speed \( V_e(\rho_e) \), this is in fact the case. Therefore, Eq. (5.39) may be considered as generalization of the equation for the speed of sound in gases or fluids. However, in traffic flows, the minus sign corresponds to the stable solution, while the plus sign corresponds to the unstable one [17]. Therefore, finite-size perturbations in traffic flow propagate backwards with respect to the average velocity \( V_e(\rho_e) \) of motion. This is in full agreement with empirical observations and overcomes a criticism of Daganzo [17].

5.1.6 Treatment of Perturbations at Fixed Cross Sections

Until now, we treated the case of a spatial variation in the density or velocity at some initial time \( t = 0 \). Analogously, however, one can treat a temporal perturbation of the stationary and homogeneous solution at a fixed location \( x = 0 \) [?, ?]. For this, one would make the ansatz

\[
\delta \rho(x, t) = \delta \rho_0 \exp \left( (\lambda + i \kappa)x - i \omega t \right), \quad \delta V(x, t) = \delta V_0 \exp \left( (\lambda + i \kappa)x - i \omega t \right).
\]

(5.41)

This does not change the density dependence of the instability area, since its boundary values are characterized by \( \lambda = 0 \) and the same characteristic polynomial. This time, however, the general solution reads

\[
\delta \rho(x, t) = \sum_l \int d\omega \delta \rho_l(\omega) \exp \left( [\lambda_l(\omega) + i \kappa_l(\omega)]x - i \omega t \right),
\]

\[
\delta V(x, t) = \sum_l \int d\omega \delta V_l(\omega) \exp \left( [\lambda_l(\omega) + i \kappa_l(\omega)]x - i \omega t \right).
\]

(5.42)

5.2 Instability in Car-Following Models and “Phantom Traffic Jams”

We have seen that macroscopic traffic models behave unstable with respect to small perturbations in a certain density range, where the average velocity changes too rapidly with the density. It would be interesting to see, whether this finding is consistent with the implications of car-following models. For this, let us study the stability of the “optimal velocity” car-following model introduced in Sec. ??.

For our analysis we imagine the situation of \( N \) vehicles \( i \) distributed over a circular road of length \( L \). This allows us to assume periodic boundary conditions. The stationary solution for this case is given by \( dV_i/dt = 0 \) and \( dd_i/dt = 0 \), which implies

\[
d_i(t) = d_o = L/N = \text{const.}
\]

\[
v_i(t) = v_{i-1}(t) = v_o(d_o) = \text{const.}
\]

(5.43)
Fig. 5.5. Illustration of the emergence of “phantom traffic jams” from initially homogeneous traffic (see the simulation applet at http://www.trafficsimulation.de). The spontaneous breakdown of traffic flow also occurs in the case of single lane traffic or identical vehicles (cars only or trucks only).

We are now interested how the deviations from this solution, i.e. the variables

\[ \delta d_i(t) = d_i(t) - d_e, \]
\[ \delta v_i(t) = v_i(t) - v_o(d_e), \]

develop in time, assuming that the initial deviations are small, i.e. \( \delta d_i(0) \ll d_e \) and \( \delta v_i(0) \ll v_o(d_e) \). For this, we linearize the model equations (2.25) and (2.26) around the stationary and homogeneous solution. This results in

\[ \frac{d\delta v_i(t)}{dt} = \frac{1}{\tau} \left[ \frac{dv_o(d_e)}{dd} \delta d_i(t) - \delta v_i(t) \right], \]
\[ \frac{d\delta d_i(t)}{dt} = \delta v_{i-1}(t) - \delta v_i(t). \]
For the analysis of stability, we use the solution ansatz
\begin{align}
\delta v_j(t) &= \delta v_0 e^{i2\pi \frac{jk}{N} + \tilde{\lambda} t}, \\
\delta d_j(t) &= \delta d_0 e^{i2\pi \frac{jk}{N} + \tilde{\lambda} t} = \delta d_0 e^{i \frac{j\kappa}{N} \frac{L}{\tilde{\lambda}} + \tilde{\lambda} t},
\end{align}
(5.46)
where \( \kappa = 2\pi k/L \) is the so-called wave number, which is inversely proportional to the wave length \( 2\pi/\kappa = L/k \). Note that, due to the assumed periodic boundary conditions, possible wavelength are fractions \( L/k \) of the length \( L \) or the circular road. The shortest wave length is given by the average vehicle distance \( d_e = L/N \), i.e. \( k \in \{1, 2, \ldots, N\} \). Summing up the functions (5.46) over these values of \( k \) results in the Fourier representation of \( \delta v_j(t) \) and \( \delta d_j(t) \):
\begin{align}
\delta v_j(t) &= \sum_{k=1}^{N} \delta v_k e^{i2\pi \frac{jk}{N} + \tilde{\lambda} t}, \\
\delta d_j(t) &= \sum_{k=1}^{N} \delta d_k e^{i2\pi \frac{jk}{N} + \tilde{\lambda} t},
\end{align}
(5.47)
where the parameters $\delta v_k$ and $\delta d_k$ are determined by the initial conditions of all vehicles $j$. $\lambda = \lambda - i\omega$ are the so-called eigenvalues, whose real part $\lambda$ describes an exponential growth (if $\lambda > 0$) or decay (if $\lambda < 0$), and whose imaginary part $\omega$ reflects oscillation frequencies. $\delta d_0$ and $\delta v_0$ denote oscillation amplitudes. Inserting this into (5.46) and dividing by $e^{i2\pi j/N+\lambda t}$, we finally obtain

$$\tilde{\lambda} \delta v_0 = 1 \left[ \frac{dv_o(d_v)}{dd} \delta d_0 - \delta v_0 \right], \quad (5.48)$$

$$\tilde{\lambda} \delta d_0 = \delta v_0 e^{-i2\pi k/N} - \delta v_0 = \delta v_0 \left( e^{-i2\pi k/N} - 1 \right). \quad (5.49)$$

Multiplying Eq. (5.48) with $\tilde{\lambda}$ and inserting Eq. (5.49) for $\tilde{\lambda} \delta d_0$ in the square brackets gives, after division by $\delta v_0$, the characteristic polynomial in the eigenvalues $\tilde{\lambda}$, namely

$$\tilde{\lambda}^2 + \frac{1}{\tau} \tilde{\lambda} - \frac{1}{\tau} \frac{dv_o(d_v)}{dd} \left( e^{-i2\pi k/N} - 1 \right) = 0. \quad (5.50)$$

The solutions $\tilde{\lambda}(k)$ of this polynomial are the eigenvalues. They read

$$\tilde{\lambda}_\pm(k) = -\frac{1}{2\tau} \pm \sqrt{\frac{1}{4\tau^2} + \frac{1}{\tau} \frac{dv_o(d_v)}{dd} \left( e^{-i2\pi k/N} - 1 \right)}. \quad (5.51)$$

Unfortunately, the square root contains a complex number, which makes it difficult to see the sign of the real value $\lambda$ of $\tilde{\lambda}$. However, considering $e^{\pm i\phi} = \cos(\phi) \pm i \sin(\phi)$ and defining the real part $\Re = \frac{1}{4\tau^2} - \frac{1}{\tau} \frac{dv_o(d_v)}{dd} [1 - \cos(2\pi k/N)] \quad (5.52)$ and its imaginary part

$$\Im = -\frac{\sin(2\pi k/N)}{\tau} \frac{dv_o(d_v)}{dd}, \quad (5.53)$$

we can again apply the very useful formula (5.27), from which we can conclude that $\lambda = \Re(\tilde{\lambda}) = 0$ if

$$\frac{1}{16\tau^4} = \frac{\Re}{4\tau^2} + \frac{\Im^2}{4} \quad (5.54)$$

(see Sec. ??). Inserting Eqs. (5.52) and (14.53), we find

$$\frac{\sin^2(2\pi k/N)}{4\tau^2} \left( \frac{dv_o(d_v)}{dd})^2 = \frac{1}{4\tau^3} \frac{dv_o(d_v)}{dd} [1 - \cos(2\pi k/N)] \right. \quad (5.55)$$

which finally results in the condition

$$\frac{dv_o(d_v)}{dd} = \frac{1 - \cos(2\pi k/N)}{\tau \sin^2(2\pi k/N)} \quad k \to 0 \frac{1}{2\tau}. \quad (5.56)$$
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The limit $2\pi k/N \to 0$ follows from $\cos(\varphi) \approx 1 - \varphi^2/2$ and $\sin(\varphi) \approx \varphi$ in the limit of small wave numbers $\kappa = 2\pi k/L$, i.e. large wave lengths $2\pi/\kappa = L/k$. Remember that the instability in our analysis of macroscopic traffic equations was largest for small wave numbers [see the paragraph after Eq. (??)]. In fact, it can be demonstrated by numerical analyses that

$$\frac{dv_o}{dd} > \frac{1}{2\tau} \tag{5.57}$$

constitutes the instability condition of the optimal velocity model. In other words, if the velocity changes too strongly with the distance, small variations of the vehicle distance or speed will grow and finally cause emergent waves, i.e. the formation of one or several traffic jams. Since the origin of such a breakdown can be infinitesimally small, these traffic jams seem to have no origin. In such situations, one speaks of “phantom traffic jams”. A closer analysis for realistic speed-distance relationships $v_o(d)$ shows that traffic tends to be unstable at medium densities $\rho = 1/d$, while it tends to be stable at small and large densities (where the speed does not change much with a variation in the distance). Only a sufficient reduction in the adaptation time $\tau$ can avoid an instability of traffic flow, while large delays in the velocity adjustment lead to growing perturbations of traffic flow.

As a setup with periodic boundary conditions is, of course, somewhat artificial, when compared to real traffic situations. Therefore, one should not conclude that “phantom traffic jams” can already explain the breakdowns of traffic flows that we observe. In fact, the densities at which traffic becomes unstable, never evolve naturally on a homogeneous freeway stretch. They only occur when additional vehicles are pumped into the system, i.e. close to onramps or intersections (see Sec. ??). Considering this, we will eventually gain a more complex, but also more realistic picture of traffic congestion and its dynamics. Nevertheless, the above stability analysis and the methods introduced will turn out to be very useful in our further considerations.

We conclude this section with an analysis of the group velocity, at which perturbations of traffic flow propagate. Considering the motion of vehicles with speed $v_o(d_e)$ with respect to a stationary reference frame, Eq. (5.27), and $\kappa = 2\pi k/L$, analogously to Eq. (5.39) one can derive

$$c_{\pm} = v_o(d_e) + \frac{\omega_{\pm}}{d\kappa} = v_o(d_e) + \frac{L}{2\pi} \frac{d\omega_{\pm}}{dk} = v_o(d_e) + \frac{L}{2\pi} \frac{d}{dk} \sqrt{\frac{1}{2} (\sqrt{\Re^2 + 3^2} - \Re)} \tag{5.58}$$

It is remarkable that the group velocity of the optimal velocity model can again have values greater than $v_o(d_e)$. However, as in the previous section, this larger group velocity is related to a negative real part $\lambda_+$ of the eigenvalue $\lambda$, i.e. it does not play a practically relevant role.
5.3 Problems

1. Derive Eq. (5.27) from the common representation of complex numbers

\[ z = |z|e^{\pm i\varphi} = |z|[\cos(\varphi) \pm i\sin(\varphi)], \tag{5.59} \]

using only the well-known trigonometric relationship \( \sin^2(x) + \cos^2(x) = 1 \) and the law \( e^{x_1} \cdot e^{x_2} = e^{x_1 + x_2} \) which holds for exponential functions. Use the circumstance that the real and imaginary parts of a complex number are orthogonal to each other.

2. Linearize the Lighthill-Whitham-Richards equation and perform the linear stability analysis. What is the real part of the complex eigenvalue of the corresponding characteristic polynomial? What are the implications for the change of the amplitude of the density deviation \( \delta\rho(x, t) \) in time? What can one say about the stability of the Burgers equation (??) containing an additional diffusion term \( D\partial^2\rho(x, t)/\partial x^2 \)?

3. Linearize the continuity equation and the velocity equation

\[
\frac{\partial V(x, t)}{\partial t} + V(x, t) \frac{\partial V(x, t)}{\partial x} = -\frac{1}{\rho(x, t)} \frac{\partial \mathcal{P}(x, t)}{\partial x} + \frac{1}{\tau} [V^e(\rho, V) - V(x, t)], \tag{5.60}
\]

in the case where the pressure \( \mathcal{P}(\rho, V) \) and the dynamic equilibrium velocity \( V^e(\rho, V) \) are not only dependent on the density \( \rho(x, t) \), but also on the average velocity \( V(x, t) \). (For simplicity, we have dropped the viscosity term, as this does not lead to qualitatively different findings.) Perform the linear stability analysis and determine the condition for unstable traffic flows. Does the speed-dependence of the traffic pressure and the dynamic equilibrium velocity increase or decrease the stability of traffic flows? How does the instability condition look like for Payne’s macroscopic traffic model, see Eq. (??)?

4. Compare the instability condition for the optimal velocity model with the one of Payne’s macroscopic traffic model. What is the interpretation of the result?

5. Assuming a Gaussian wave packet

\[ \delta\rho^I(\kappa) = \frac{e^{-(\kappa-\kappa_0)^2/(2\theta)}}{\sqrt{2\pi\theta}} \tag{5.61} \]

and a given dependency \( \omega_l(\kappa) \) between wave number \( \kappa \) and frequency \( \omega_l \), show that the superposition behaves like a wave with frequency \( \omega_l(\kappa_0) \) and speed \( \omega_l(\kappa_0)/\kappa_0 \), but its amplitude is moving with the group velocity \( c_l = d\omega_l(\kappa)/d\kappa \). What relationship exists between the variance of wave numbers and the variance of the amplitude in space?
5.4 Further Reading


Non-Linear Instability and Other Non-Linear Effects

It is interesting to ask what happens for densities that are just a little bit “left” or “right” of the linearly unstable density regime defined by the condition (5.57). Is traffic at these densities unconditionally stable? Or does stability depend on the size of perturbation of the traffic flow? That is, would perturbations with a subcritical amplitude fade away, while “supercritical” perturbations would increase and cause a traffic jam? In other words, would the instability regime (i.e. the density range of unstable traffic) depend on the amplitude of perturbations in the traffic flow?

To answer these questions, we need to study deviations from the stationary and homogeneous solutions which are not anymore small, i.e. we basically have to perform a non-linear stability analysis. Unfortunately, this brings us to equations which are hardly tractable. The positive side of it is the possibility to learn about exact solution methods for differential equations and approximate methods from perturbation theory.

For simplicity, let us again start from the optimal velocity model of Sec. ??, but this time, carry out a second order Taylor approximation rather than a first order one. We will use the formula

\[
\left[ f(x + \Delta x) \right]_n \approx f(x) + \frac{df(x)}{dx} \Delta x + \frac{d^2 f(x)}{dx^2} \frac{(\Delta x)^2}{2} + \cdots = \sum_{k=0}^{n} \frac{d^k f(x)}{dx^k} \frac{(\Delta x)^k}{k!} \quad (6.1)
\]

for the \( n \)-th order Taylor approximation of a sufficiently well-behaved function \( f(x) \), where

\[
k! = k \cdot (k - 1) \cdot (k - 2) \ldots 2 \cdot 1 . \quad (6.2)
\]

To second order \((k = 2)\), from Eq. (2.25) we obtain

\[
\frac{d\delta \nu_j(t)}{dt} = \frac{1}{\tau} \left[ \frac{d\nu_o(d_e)}{dd} \delta d_j(t) + \frac{1}{2} \frac{d^2 \nu_o(d_e)}{dd^2} [\delta d_j(t)]^2 - \delta \nu_j(t) \right], \quad (6.3)
\]

\[
\frac{d\delta d_j(t)}{dt} = \delta \nu_{j-1}(t) - \delta \nu_j(t). \quad (6.4)
\]
By deriving Eq. (6.4) with respect to $t$, inserting (6.3) for $\delta v_j(t)$ and $\delta v_{j-1}(t)$, considering $\delta v_{j-1}(t) - \delta v_j(t) = d\delta d_j/dt$, and applying the abbreviations

$$x_j(t) = \delta d_j(t), \quad \gamma = \frac{1}{2\tau}, \quad (\omega_0)^2 = \frac{1}{\tau} \frac{d v_o}{dr}, \quad \text{and} \quad B = \frac{1}{2\tau} \frac{d^2 v_o}{dr^2},$$

we get the following non-linear system of ordinary differential equations of second order with constant coefficients:

$$\frac{d^2 x_j(t)}{dt^2} + 2\gamma \frac{dx_j(t)}{dt} + (\omega_0)^2 x_j(t) = (\omega_0)^2 x_{j-1}(t) + \epsilon B \left\{ [x_{j-1}(t)]^2 - [x_j(t)]^2 \right\}. \quad (6.5)$$

Here, we have introduced a prefactor $\epsilon$, which allows us to turn on (for $\epsilon = 1$) or off (for $\epsilon = 0$) the non-linear term. In this way, we can treat the last term as perturbation of the remaining linear system of ordinary differential equations. Looking at the form of Eq. (6.6), one can see that this system is composed of coupled equations for externally driven, damped harmonic oscillators [?].

### 6.1 Perturbation Theoretical Approach

We may expand the solution $x_j(t)$ into contributions of order $\epsilon$ in the following way:

$$x_j(t) = X_j(t) + \epsilon Y_j(t) + \epsilon^2 Z_j(t) + \ldots \quad (6.7)$$

We are looking for a solution which remains valid when the perturbation size $\epsilon$ is varied between 0 and 1. As the polynomials 1, $\epsilon$, and $\epsilon^2$ form a linearly independent set of functions in $\epsilon$ [?], the prefactors belonging to each order of $\epsilon$ must be zero, which results in the following hierarchy of coupled and inhomogeneous, but linear differential equations:

$$\frac{d^2 X_j(t)}{dt^2} + \epsilon \frac{d^2 Y_j(t)}{dt^2} + \epsilon^2 \frac{d^2 Z_j(t)}{dt^2}$$

$$+ 2\gamma \frac{d X_j(t)}{dt} + 2\epsilon \gamma \frac{d Y_j(t)}{dt} + 2\epsilon^2 \gamma \frac{d Z_j(t)}{dt}$$

$$+ (\omega_0)^2 X_j(t) + \epsilon (\omega_0)^2 Y_j(t) + \epsilon^2 (\omega_0)^2 Z_j(t)$$

$$= (\omega_0)^2 X_{j-1}(t) + \epsilon (\omega_0)^2 Y_{j-1}(t) + \epsilon^2 (\omega_0)^2 Z_{j-1}(t)$$

$$+ \epsilon B [X_{j-1}(t) + \epsilon Y_{j-1}(t) + \epsilon^2 Z_{j-1}(t)]^2$$

$$- \epsilon B [X_j(t) + \epsilon Y_j(t) + \epsilon^2 Z_j(t)]^2. \quad (6.8)$$

We are looking for a solution which remains valid when the perturbation size $\epsilon$ is varied between 0 and 1. As the polynomials 1, $\epsilon$, and $\epsilon^2$ form a linearly independent set of functions in $\epsilon$ [?], the prefactors belonging to each order of $\epsilon$ must be zero, which results in the following hierarchy of coupled and inhomogeneous, but linear differential equations:
6.2 Solution of the Homogeneous System of Linear Differential Equations

\[ \frac{d^2 X_j(t)}{dt^2} + 2\gamma \frac{dX_j(t)}{dt} + (\omega_0)^2 X_j(t) = (\omega_0)^2 X_{j-1}(t), \quad (6.9) \]

\[ \frac{d^2 Y_j(t)}{dt^2} + 2\gamma \frac{dY_j(t)}{dt} + (\omega_0)^2 Y_j(t) = (\omega_0)^2 Y_{j-1}(t) \]

\[ + B \left\{ [X_{j-1}(t)]^2 - [X_j(t)]^2 \right\}, \quad (6.10) \]

\[ \frac{d^2 Z_j(t)}{dt^2} + 2\gamma \frac{dZ_j(t)}{dt} + (\omega_0)^2 Z_j(t) = (\omega_0)^2 Z_{j-1}(t) \]

\[ + 2B \left[ X_{j-1}(t)Y_{j-1}(t) - X_j(t)Y_j(t) \right]. \quad (6.11) \]

Obviously, these sets of differential equations must be solved subsequently, i.e. the solution of Eq. (6.9) must be inserted into Eq. (6.10) to solve it afterwards, etc. That is, the equations for higher-order corrections in \( \epsilon \) depend only on the solutions of the equations determining corrections of lower order in \( \epsilon \). For this reason, we have used the notion “hierarchy” of equations above. We conclude that a system of non-linear equations may be transformed into a hierarchy of systems of linear equations.

6.2 Solution of the Homogeneous System of Linear Differential Equations

Let us begin with solving the set of differential equations (6.9). It is common to solve the homogeneous differential equations

\[ \frac{d^2 X_j(t)}{dt^2} + 2\gamma \frac{dX_j(t)}{dt} + (\omega_0)^2 X_j(t) = 0 \quad (6.12) \quad \{\text{homdi}\} \]

first, where the right-hand side has been set to zero. In this way, we get rid of the coupling between the equations for different indices (vehicles) \( j \). Inserting the ansatz

\[ X_j(t) = X_j^0 e^{\tilde{\lambda} t} \quad (6.13) \quad \{\text{ansatz1}\} \]

results in the second-order characteristic polynomial

\[ \tilde{\lambda}^2 + 2\gamma \tilde{\lambda} + (\omega_0)^2 = 0 , \quad (6.14) \]

which is solved by the two eigenvalues

\[ \tilde{\lambda}_\pm = \lambda_\pm = -\gamma \pm \sqrt{\gamma^2 - (\omega_0)^2}. \quad (6.15) \]

If \( \gamma^2 > (\omega_0)^2 \), i.e. \( dv/(dv)/dd < 1/(4\tau) \), the general solution of the homogeneous differential equation is just a superposition of two decaying exponential functions:

\[ X_j^{\text{hom}}(t) = X_j^+ e^{\tilde{\lambda}_+ t} + X_j^- e^{\tilde{\lambda}_- t}. \quad (6.16) \quad \{\text{homsol}\} \]

\( X_j^+ \) and \( X_j^- \) are determined by the initial conditions \( X_j(0) \) and \( dX_j(0)/dt \).
In the case \((ω_0)^2 > γ^2\), \(dv_e(du)/dd > 1/(4τ)\), we can write
\[
\tilde{λ}_± = λ - iω_± = -γ ± i\sqrt{(ω_0)^2 - γ^2},
\]
\[\text{(6.17)}\] \(\text{and1}\)
i.e. \(λ = -γ\) and \(ω_± = ±\sqrt{(ω_0)^2 - γ^2}\). Because of \(e^{±iωt} = \cos(ωt) ± i\sin(ωt)\), the solution \((6.18)\) can then also be written in the form
\[
X_j^{\text{hom}}(t) = \left[ X_j^1 \cos(ωt) + X_j^2 \sin(ωt) \right] e^{-γt},
\]
\[\text{(6.18)}\] \(\text{homsol}\)
where \(X_j^1\) and \(X_j^2\) are determined by the initial conditions \(X_j(0)\) and \(dX_j(0)/dt\). This solution with exponentially decaying cosine and sine waves is more interesting for us as we know that, for \(dv_e(du)/dd > 1/(2τ)\), traffic behaves unstable. In other words, small perturbations are growing although \(X_j^{\text{hom}}(t)\) is exponentially decaying in time. We call this phenomenon \(\text{convective instability}\). As we will see below, it is based on a \(\text{resonance effect}\) known for externally driven, damped harmonic oscillators. In order to show this, we will have to solve the full inhomogeneous Eq. \((6.9)\).

### 6.3 Solution of the Inhomogeneous System of Linear Differential Equations

Let us introduce the abbreviation \(U_j(t) = dX_j(t)/dt\) and rewrite the second order differential equation as two coupled differential equations of first order. In matrix representation, we have
\[
\begin{aligned}
\frac{d}{dt} \begin{pmatrix} X_j(t) \\ U_j(t) \end{pmatrix} &= \begin{pmatrix} 0 & 1 \\ -(ω_0)^2 & -2γ \end{pmatrix} \begin{pmatrix} X_j(t) \\ U_j(t) \end{pmatrix} + \begin{pmatrix} 0 \\ (ω_0)^2X_{j-1}(t) \end{pmatrix} .
\end{aligned}
\]
\[\text{(6.19)}\]

From Eqs. \((??)\) and \((6.17)\) we can conclude that the matrix
\[
Φ(t) = \begin{pmatrix} e^{λ_+t} & e^{λ_-t} \\ λ_+ e^{λ_+t} & λ_- e^{λ_-t} \end{pmatrix}
\]
\[\text{(6.20)}\]
is a fundamental system of the homogeneous set of differential equations (corresponding to \(X_{j-1}(t) = 0\)). The inverse is
\[
Φ^{-1}(t) = \frac{1}{\det(Φ)} \begin{pmatrix} λ_- e^{λ_-t} & -e^{λ_-t} \\ -λ_+ e^{λ_+t} & e^{λ_+t} \end{pmatrix}
\]
\[\text{(6.21)}\]
(see Exercise .........) with
\[
\det(Φ) = (λ_- - λ_+)e^{(λ_- + λ_+)t}.
\]
\[\text{(6.22)}\]
A special solution of the inhomogeneous system of differential equations (6.19) can then be calculated via the formula

\[
\begin{pmatrix}
X_j^{inh}(t) \\
U_j^{inh}(t)
\end{pmatrix}
= \Phi(t) \int_0^t dt' \Phi^{-1}(t') \begin{pmatrix} 0 \\ (\omega_0)^2 X_{j-1}(t') \end{pmatrix}.
\]

(6.23)

Assuming \(X_j(t) = X_j^0 e^{\lambda t}\), one eventually finds

\[
\begin{pmatrix}
X_j^{inh}(t) \\
U_j^{inh}(t)
\end{pmatrix} = \left( \begin{array}{cc}
\tilde{\lambda}_+ e^{\tilde{\lambda}_+ t} & e^{\tilde{\lambda}_- t} \\
\tilde{\lambda}_- e^{\tilde{\lambda}_- t} & e^{\tilde{\lambda}_+ t}
\end{array} \right) \begin{pmatrix}
-\frac{X_j^{inh}(0) + (\omega_0)^2}{(\tilde{\lambda}_+ - \tilde{\lambda}_-)(\lambda - \lambda_+)} e^{\tilde{\lambda}_+ t} - (\lambda - \tilde{\lambda}_+) e^{\tilde{\lambda}_- t} \\
\frac{X_j^{inh}(0) + (\omega_0)^2}{(\tilde{\lambda}_+ - \tilde{\lambda}_-)(\lambda - \lambda_-)} e^{\tilde{\lambda}_+ t} - (\lambda - \tilde{\lambda}_+) e^{\tilde{\lambda}_- t}
\end{pmatrix}.
\]

(6.24)

This finally results in the special solution of the inhomogeneous system of differential equations

\[
X_j^{inh}(t) = X_j^{inh}(0) + (\omega_0)^2 \frac{(\tilde{\lambda}_+ - \tilde{\lambda}_-) e^{\tilde{\lambda}_+ t} + (\tilde{\lambda}_- - \tilde{\lambda}_+) e^{\tilde{\lambda}_- t} - (\lambda - \tilde{\lambda}_+) e^{\tilde{\lambda}_- t}}{(\tilde{\lambda}_+ - \tilde{\lambda}_-)(\lambda - \lambda_+)}.
\]

(6.25) \{inhomsol1\}

The general solution is given by

\[
X_j(t) = C_1 X_j^{hom}(t) + C_2 X_j^{inh}(t),
\]

(6.26) \{inhomsol2\}

where the constants \(C_1\) and \(C_2\) are given by the initial conditions \(X_j(0)\) and \(dX_j(0)/dt\). Considering the fact that both eigenvalues \(\tilde{\lambda}_\pm\) have a negative real part \(-\gamma\) corresponding to an exponential decay, we can state

\[
\lim_{t \to \infty} X_j(t) = \lim_{t \to \infty} X_j^{inh}(t) = \frac{X_j^{inh}(0) + (\omega_0)^2 e^{\tilde{\lambda}_+ t}}{(\lambda - \tilde{\lambda}_+)(\tilde{\lambda}_+ - \tilde{\lambda}_-)} = \frac{X_j^{inh}(0) + (\omega_0)^2 e^{\tilde{\lambda}_+ t}}{\tilde{\lambda}_+^2 + 2\gamma \tilde{\lambda}_+ + (\omega_0)^2}.
\]

(6.27)

### 6.4 Derivation of the Instability Condition of the Linear System of Differential Equations

Now, let us assume \(\tilde{\lambda} = i\alpha\), i.e. a periodic variation \(X_{j-1}(t) = X_j^0 e^{i\alpha t} = X_j^0 \left[ \cos(\alpha t) + i \sin(\alpha t) \right]\) of the preceding vehicle \(j - 1\), as expected in the marginally stable case with no increase or decrease in the amplitude. In this case,

\[
\lim_{t \to \infty} X_j(t) = \frac{X_j^{inh}(0) + (\omega_0)^2 e^{\tilde{\lambda}_+ t}}{-\alpha^2 + 2i\alpha + (\omega_0)^2} = \frac{X_j^{inh}(0) + (\omega_0)^2 e^{i\alpha t}}{\sqrt{[(\omega_0)^2 - \alpha^2]^2 + (2\alpha \gamma)^2 e^{i\alpha t}}}.
\]

(6.28)
with
\[
\tan \varphi = \frac{\sin \varphi}{\cos \varphi} = \frac{2\alpha \gamma}{(\omega_0)^2 - \alpha^2} \quad (6.29)
\]
(see Fig. ??). Consequently, the proportionality factor between both oscillations is
\[
\lim_{t \to \infty} \frac{X_j(t)}{X_{j-1}e^{\alpha t}} = \frac{(\omega_0)^2 e^{-i\varphi}}{\sqrt{[(\omega_0)^2 - \alpha^2]^2 + (2\alpha \gamma)^2}} = F e^{-i\varphi}. \quad (6.30)
\]
The amplification factor \( F \) and the phase shift \(-\varphi\) with respect to the oscillation of the previous vehicle \( j - 1 \) imply a different effective instability behavior. In the limit \( t \to \infty \), the inhomogeneous set of differential equations (14.11) may be rewritten as
\[
\frac{d^2 X_j(t)}{dt^2} + 2\gamma \frac{dX_j(t)}{dt} + (\omega_0)^2 \left(1 - \frac{e^{i\varphi}}{F}\right) X_j(t) = 0 \quad (6.31)
\]
i.e. the parameter \((\omega_0)^2\) in the homogeneous system of differential equations (6.12) is effectively replaced by
\[
(\omega'_0)^2 = (\omega_0)^2 \left(1 - \frac{e^{i\varphi}}{F}\right). \quad (6.32)
\]
Note that the time delay (or travelling wave) representation (6.31) effectively decouples the dynamics of neighboring vehicles, but the real parts of the eigenvalues and, therefore, the stability change compared to the solution of the similarly looking homogeneous system of linear differential equations (??).

Due to \(|e^{-i\varphi}| = 1\), the absolute value of the proportionality factor, the amplification factor \( F \), is greater than 1 if
\[
(\omega_0)^4 > [(\omega_0)^2 - \alpha^2]^2 + (2\alpha \gamma)^2 = (\omega_0)^4 - 2\alpha^2(\omega_0)^2 + \alpha^4 + 4\alpha^2 \gamma^2 \quad (6.33)
\]
or \(2(\omega_0)^2 > \alpha^2 + 4\gamma^2\). Considering the abbreviations (6.5), this implies
\[
\frac{dv_e(d_0)}{dd} > \frac{\alpha^2 \tau}{2} + \frac{1}{2\tau}. \quad (6.34)
\]
If we consider that \(\alpha\) may have any value including \(\alpha = 0\), the condition for possible amplification becomes \(dv_e(d_0)/dd > 1/(2\tau)\), which exactly agrees with the instability condition (5.57). Our new calculation shows that the instability can result due to a resonance effect, i.e. even if the oscillations are decreasing in time due to \(\lambda = \text{Re}(\tilde{\lambda}) = -\gamma\), they increase from one car \(j - 1\) to the next car \(j\), if the amplitude grows due to a so-called resonance effect. This is the case, if the instability condition (??) is fulfilled. We will need these findings again in Sec. ??.
6.5 Solutions with Stable Amplitudes

Let us now investigate the conditions under which the amplitude of oscillation stays exactly the same, i.e. \( F = 1 \). In this case, we can derive from Eq. (6.33) a condition for the driving frequency \( \alpha \), namely

\[ \alpha^2 = 2(\omega_0)^2 - 4\gamma^2. \] (6.35)

Inserting \( \alpha = \sqrt{2(\omega_0)^2 - 4\gamma^2} \) into Eq. (6.29), we find

\[ \tan(\varphi) = \frac{2\gamma\sqrt{2(\omega_0)^2 - 4\gamma^2}}{4\gamma^2 - (\omega_0)^2} = \frac{\sqrt{(\omega_0)^2 - 1}}{1 - \frac{(\omega_0)^2}{4\gamma^2}}. \] (6.36) \{finimp\}

This defines a phase shift \( \varphi \) which is consistent with no amplification or decay. Interestingly enough, the square root is positive exactly when the linear instability condition (5.57) is fulfilled, which may be written as \( (\omega_0)^2 > 2\gamma^2 \).

The same result as Eq. (6.36) can be obtained from the condition (??) for marginally stable oscillations of vehicles on a circular road. The equivalent relationship

\[ 1 - \cos(2\pi k/N) = \tau \frac{d\nu_o(d_o)}{dd} = \frac{(\omega_0)^2}{4\gamma^2}. \] (6.37) \{rewri\}

Expressing \( \sin^2(\varphi) \) and \( \cos(\varphi) \) by \( \tan(\varphi) \) (see Exercise 5...), and using the abbreviation \( y = 1 + \tan^2(2\pi k/N) \), allows us to rewrite Eq. (6.37) as

\[ \frac{1 - \cos(2\pi k/N)}{\sin^2(2\pi k/N)} = \frac{1 \mp 1/\sqrt{y}}{(y^2 - 1)/y^2} = \frac{y \mp \sqrt{y}}{y - 1} = \frac{\sqrt{y}}{\sqrt{y} \pm 1}. \] (6.38)

With Eq. (6.37) we get

\[ \sqrt{y} = \frac{(\omega_0)^2}{4\gamma^2} (\sqrt{y} \pm 1) \quad \text{or} \quad \sqrt{y} = \frac{\pm(\omega_0)^2/(4\gamma^2)}{1 - (\omega_0)^2/(4\gamma^2)}, \] (6.39)

which finally implies Eq. (6.36) with \( \varphi = 2\pi k/N \). Consequently, \( N \) times the phase shift \( \varphi \) must be a multiple of \( 2\pi \). (The phase shift of the \( N \)-th vehicle modulo \( 2\pi \) must be 0 on a ring road, as it reflects the phase shift between the motion of the \( j \)-th and the \( j + N \)-th vehicle, which are the same.) Considering \( k \in \{1, 2, \ldots, N\} \), this suggests that it may be difficult to establish marginal stability in the linearized optimal velocity model, if the number \( N \) of vehicles is small (see Exercise 5...). We should, however, keep in mind that the non-linear contributions in the original model are expected to stabilize the amplitudes at finite values, if the linearized system behaves linearly unstable.
6.6 Calculation of Perturbative Corrections

One advantage of treating interacting vehicles as forced damped oscillators is the possibility to study string instability on roads that are not circular (see Exercise 5....). Here, we assume that one of the vehicles performs periodic variations around the equilibrium distance and speed at frequency $\alpha$. In the limit $t \to \infty$, we get

$$X_j(t) = X_{j-1}(t) e^{-i\varphi} \quad \text{and} \quad (X_j(t))^2 = (X_{j-1}(t))^2 F^2 e^{-i2\varphi} \quad (6.40)$$

with $X_{j-1}(t) = X_{j-1}^0 e^{i\omega t}$. This can be inserted into Eq. (6.9) in order to give

$$\frac{d^2 Y_j(t)}{dt^2} + 2\gamma \frac{dY_j(t)}{dt} + (\omega_0)^2 Y_j(t) = (\omega_0)^2 Y_{j-1}(t)$$

$$- B \left( F^2 e^{-i2\varphi} - 1 \right) \cdot (X_{j-1}^0)^2 e^{2i\omega t}. \quad (6.41)$$

Assuming $Y_{j-1}(t) = 0$, this equation can be solved by means of Eqs. (6.18), (6.25), and (6.26). Analogously to Eq. (6.27), with $\tilde{\lambda} = i2\alpha$ we find

$$\lim_{t \to \infty} Y_j(t) = B \left( F^2 e^{-i2\varphi} - 1 \right) \frac{(X_{j-1}^0)^2 e^{i2\omega t}}{-4\alpha^2 - 2i\alpha(\tilde{\lambda}_+ + \tilde{\lambda}_-) + \tilde{\lambda}_+ \tilde{\lambda}_-}$$

$$= B \left( F^2 e^{-i2\varphi} - 1 \right) \frac{(X_{j-1}^0)^2 e^{i2\omega t} e^{-i\beta}}{\sqrt{[(\omega_0)^2 - 4\alpha^2]^2 + (4\alpha\gamma)^2}} \quad (6.42)$$

with

$$\tan \beta = \frac{4\alpha\gamma}{(\omega_0)^2 - 4\alpha^2}, \quad (6.43)$$

where we have used the relationships $\tilde{\lambda}_+ \tilde{\lambda}_- = (\omega_0)^2$ and $\tilde{\lambda}_+ + \tilde{\lambda}_- = -2\gamma$, see Eq. (??). Note that the corrections due to non-linear interactions are characterized by multiple wave frequencies (which is particularly well-known from the sound of music instruments).

According to Eq. (??), we have $x_j(t) \approx X_j(t) + eY_j(t)$. Looking at this improved solution, we make two interesting observations: The amplitude of the correction $Y_j(t)$ depends on the perturbation amplitude $X_{j-1}^0$. Moreover, it oscillates with twice the frequency of the external forcing $x_{j-1}(t) = Y_{j-1}(t)$. As a consequence, the vehicle $j + 1$ is expected to show oscillations of even higher frequency. Therefore, the question is, how the solution would look like if we would solve the non-linear set of differential equations (??) on a circular road, where each vehicle oscillation influences all the others. This problem will be addressed in the next section.
6.7 Fourier Representation of the Non-Linear Solution for A Circular Road

In computer simulations of identical vehicles moving on a circular one-lane road, one can see travelling waves forming in the unstable density regime. Starting with small perturbations, one would end up with a typical periodic wave profile of finite amplitude due to the non-linearities in the equations (see Sec. ??). Because of the circularity of the road, the wave profile must necessarily repeat in space after distances of $L$, i.e. every $N$ vehicles, and once a stable travelling wave is established, it moves at constant speed, which also implies periodicity in time (see Sec. ??). We will denote this period by $T$ and define the frequency $\omega = 2\pi/T$ by it.

Therefore, if we are not interested here in the dynamics of the formation of this travelling wave, but instead in the description of its spatio-temporal dependence once it has formed, we can utilize its periodicity to represent it by a Fourier series as any periodic function: However, it is not advised to use the complex series expansion

$$ x_j(t) = \sum_{-\infty}^{\infty} z_{j,n} e^{i \omega t} \quad \text{with} \quad z_{j,n} = \frac{1}{T} \int_{0}^{T} dt x_j(t) e^{-i \omega t} , \quad (6.44) $$

as for non-linear equations, one can usually not construct a real-valued solution by superposition of two conjugate complex-valued ones, as this is the case for linear equations. Therefore, we apply the real-valued series expansion

$$ x_j(t) = \frac{1}{2} a_{j,0} + \sum_{n=1}^{\infty} a_{j,n} \cos(n \omega t) + \sum_{n=1}^{\infty} b_{j,n} \sin(n \omega t) \quad (6.45) \{\text{fourdef}\} $$

with the Euler-Fourier coefficients

$$ a_{j,n} = \frac{2}{T} \int_{0}^{T} dt x_j(t) \cos(n \omega t) \quad \text{and} \quad b_{j,n} = \frac{2}{T} \int_{0}^{T} dt x_j(t) \sin(n \omega t) . \quad (6.46) $$

Note that $a_{j,0}/2$ is the average value of $x_j(t)$ over a time period $T$. This average must vanish, as the average of the distance $d_j(t) = d_e + \delta d_j(t) = d_e + x_j(t)$ would otherwise be $d_e + a_{j,0}/2$. Therefore, we can set $a_{j,0} = 0$.

A travelling wave implies that vehicle $j$ will experience the same dynamics as the respective leading vehicle $j-1$, but with a phase shift $\delta$. This implies $x_j(t) = x_{j-1}(t - \delta)$ and
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\[
x_{j-1}(t) = \sum_{n=1}^{\infty} a_{j-1,n} \cos(n\omega t) + \sum_{n=1}^{\infty} b_{j-1,n} \sin(n\omega t)
\]

\[
= \sum_{n=1}^{\infty} a_{j,n} \cos(n\omega (t + \delta)) + \sum_{n=1}^{\infty} b_{j,n} \sin(n\omega (t + \delta))
\]

\[
= \sum_{n=1}^{\infty} a_{j,n} \left[ \cos(n\omega \delta) \cos(n\omega t) - \sin(n\omega \delta) \sin(n\omega t) \right]
\]

\[
+ \sum_{n=1}^{\infty} b_{j,n} \left[ \cos(n\omega \delta) \sin(n\omega t) + \sin(n\omega \delta) \cos(n\omega t) \right]. 
\]

(6.47)

Here, we have applied the sum and difference formulas

\[
\cos(\varphi_1 \pm \varphi_2) = \cos(\varphi_1) \cos(\varphi_2) \mp \sin(\varphi_1) \sin(\varphi_2), \quad (6.48)
\]

\[
\sin(\varphi_1 \pm \varphi_2) = \sin(\varphi_1) \cos(\varphi_2) \pm \cos(\varphi_1) \sin(\varphi_2) \quad (6.49)
\]

(see Exercise 6...). Altogether, for \( n \geq 1 \) we find the relations

\[
a_{j-1,n} = a_{j,n} \cos(n\omega \delta) + b_{j,n} \sin(n\omega \delta)
\]

\[
b_{j-1,n} = b_{j,n} \cos(n\omega \delta) - a_{j,n} \sin(n\omega \delta) \quad (6.50)
\]

between the Euler-Fourier coefficients of successive vehicles. The propagation speed of the corresponding travelling wave is given by the distance \(-d_e\) travelled within the time delay \( \delta \), i.e.

\[
c = \frac{-d_e}{\delta} = -\frac{L}{N \delta}. \quad (6.51)
\]

We will now determine the series expansion of \([x_j(t)]^2\), assuming convergence of all series involved, so we can change the order of summation. Multiplying Eq. (6.45) with itself and applying the sum and difference formulas (6.48) and (6.49), we obtain the following (see also Exercise 6...):
\[ [x_j(t)]^2 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{j,m} a_{j,n} \cos(m \omega t) \cos(n \omega t) \]
\[ + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{j,m} b_{j,n} \cos(m \omega t) \sin(n \omega t) \]
\[ + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{j,m} a_{j,n} \sin(m \omega t) \cos(n \omega t) \]
\[ + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} b_{j,m} b_{j,n} \sin(m \omega t) \sin(n \omega t) \]
\[ = \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (a_{j,m} a_{j,n} - b_{j,m} b_{j,n}) \cos((m + n) \omega t) \]
\[ + \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (a_{j,m} b_{j,n} + b_{j,m} a_{j,n}) \sin((m + n) \omega t) \]
\[ + \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (a_{j,m} a_{j,n} + b_{j,m} b_{j,n}) \cos((m - n) \omega t) \]
\[ + \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (b_{j,m} a_{j,n} - a_{j,m} b_{j,n}) \sin((m - n) \omega t) \].

(6.52)

Our goal is to collect terms of the same frequency \( m \omega \) by rearrangement of the above sums. For this, we apply the identities

\[ \left( \sum_{m=1}^{\infty} a_m \right) \cdot \left( \sum_{n=1}^{\infty} b_n \right) = \sum_{m=1}^{\infty} \sum_{n=1}^{m-1} a_n b_{m-n} \]
\[ = \sum_{m=1}^{\infty} a_m b_m + \sum_{m=1}^{\infty} \sum_{n=1}^{m} a_{m+n} b_n + \sum_{m=1}^{\infty} \sum_{n=1}^{m} a_n b_{m+n} \]  

(6.53)

(6.54)

(see Exercise 6...). With these, \( \cos(-m \omega t) = -\cos(m \omega t) \), \( \sin(-m \omega t) = -\sin(m \omega t) \), \( \cos(0 \omega t) = 1 \), and \( \sin(0 \omega t) = 0 \) we get
\[ [x_j(t)]^2 = \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{m-1} (a_{j,n} a_{j,m-n} - b_{j,n} b_{j,m-n}) \cos(m\omega t) \]
\[ + \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{m-1} (a_{j,n} b_{j,m-n} + b_{j,n} a_{j,m-n}) \sin(m\omega t) \]
\[ + \frac{1}{2} \sum_{m=1}^{\infty} (a_{j,m} a_{j,m} + b_{j,m} b_{j,m}) \cos(0\omega t) \]
\[ + \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (a_{j,m} a_{j,m+n} + b_{j,m+n} b_{j,m}) \cos(m\omega t) \]
\[ + \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (a_{j,m+n} a_{j,n} + b_{j,m+n} b_{j,n}) \cos(-m\omega t) \]
\[ + \frac{1}{2} \sum_{m=1}^{\infty} (a_{j,m} a_{j,m} + b_{j,m} b_{j,m}) \sin(0\omega t) \]
\[ + \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (b_{j,m+n} a_{j,n} - a_{j,m+n} b_{j,n}) \sin(m\omega t) \]
\[ + \frac{1}{2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (b_{j,n} a_{j,m+n} - a_{j,n} b_{j,m+n}) \sin(-m\omega t). \] (6.55)

This can be simplified to give
\[ [x_j(t)]^2 = \sum_{m=1}^{\infty} A_{j,m} \cos(m\omega t) + \sum_{m=1}^{\infty} B_{j,m} \sin(m\omega t) \] (6.56)

with
\[ A_{j,m} = \frac{1}{2} \sum_{n=1}^{m-1} (a_{j,n} a_{j,m-n} - b_{j,n} b_{j,m-n}) \]
\[ + \sum_{n=1}^{\infty} (a_{j,m+n} a_{j,n} + b_{j,m+n} b_{j,n}), \]
\[ B_{j,m} = \frac{1}{2} \sum_{n=1}^{m-1} (a_{j,n} b_{j,m-n} + b_{j,n} a_{j,m-n}) \]
\[ + \sum_{n=1}^{\infty} (b_{j,m+n} a_{j,n} - a_{j,n} b_{j,m+n}). \] (6.57)

The expression for \( x_{j-1}(t) \) look the same, but with \( t \) replaced by \( t + \delta \). Considering the sum and difference formulas (6.48) and (6.49), we find \( A_{j-1,0} = A_{j,0} \) and
\[ A_{j-1,m} = A_{j,m} \cos(m \omega \delta) + B_{j,m} \sin(m \omega \delta), \]
\[ B_{j-1,m} = B_{j,m} \cos(m \omega \delta) - A_{j,m} \sin(m \omega \delta). \]  

(6.58)

This implies
\[
[x_j(t)]^2 - [x_{j-1}(t)]^2 = \sum_{m=1}^{\infty} \left[ A_{j,m} (1 - \cos(m \omega \delta)) - B_{j,m} \sin(m \omega \delta) \right] \cos(m \omega t) + \sum_{m=1}^{\infty} \left[ B_{j,m} (1 - \cos(m \omega \delta)) + A_{j,m} \sin(m \omega \delta) \right] \sin(m \omega t).
\]

(6.59)

As the cosine functions \( \cos(m \omega t) \) and the sine functions \( \sin(m \omega t) \) with \( m \in \{1, 2, \ldots \} \) are linearly independent, we have now to insert (2.41) into Eq. (6.6), carry out the time derivatives, insert Eqs. (6.58) and (6.59) and set the prefactors of cosine and sine functions of different frequencies equal to zero. Considering \( \frac{d \cos(m \omega t)}{dt} = -m \omega \sin(m \omega t) \) and \( \frac{d \sin(m \omega t)}{dt} = m \omega \cos(m \omega t) \), we eventually find for \( m \geq 1 \) the conditions
\[
-a_{j,m} m^2 \omega^2 + 2 \gamma b_{j,m} m \omega + (\omega_0)^2 \left[ a_{j,m} (1 - \cos(m \omega \delta)) + b_{j,m} \sin(m \omega \delta) \right] = -B \left[ A_{j,m} (1 - \cos(m \omega \delta)) - B_{j,m} \sin(m \omega \delta) \right]
\]

(6.60)

from the prefactors of \( \cos(m \omega t) \) and the conditions
\[
-b_{j,m} m^2 \omega^2 - 2 \gamma a_{j,m} m \omega + (\omega_0)^2 \left[ b_{j,m} (1 - \cos(m \omega \delta)) + a_{j,m} \sin(m \omega \delta) \right] = -B \left[ B_{j,m} (1 - \cos(m \omega \delta)) + A_{j,m} \sin(m \omega \delta) \right]
\]

(6.61)

from the prefactors of \( \sin(m \omega t) \).

Let us first study the limit \( \epsilon = 0 \), in which the non-linear term drops out. In order to see whether the above formulas are consistent with our previous calculations, we start with \( m = 1 \). Dividing Eq. (6.60) by \( a_{j,m} \) and (6.58) by \( b_{j,m} \) gives two different expressions for \( \omega^2 \), which must be equal to each other. Using the abbreviation \( \Delta = \omega \delta \), this implies the condition
\[
\frac{2 \gamma b_{j,1}}{a_{j,1}} \omega + (\omega_0)^2 \left[ (1 - \cos(\Delta)) - \frac{b_{j,1}}{a_{j,1}} \sin(\Delta) \right] = -\frac{2 \gamma a_{j,1}}{b_{j,1}} \omega + (\omega_0)^2 \left[ (1 - \cos(\Delta)) + \frac{a_{j,1}}{b_{j,1}} \sin(\Delta) \right]
\]

(6.62)

or
\[
\omega = \frac{(\omega_0)^2}{2 \gamma} \sin(\Delta).
\]

(6.63)
This determines the frequency $\omega$ as a function of the phase shift $\delta$ or speed $c = -L/(N\delta)$. The phase shift itself can be determined by inserting this into Eq. (6.60) for $m = 1$. In this way we get

$$0 = -a_{j,1} \frac{(\omega_0)^4}{4\gamma^2} \sin^2(\omega\delta) + b_{j,1} (\omega_0)^2 \sin(\omega\delta)$$

or

$$\frac{(\omega_0)^2}{4\gamma^2} \sin^2(\omega\delta) = (1 - \cos(\omega\delta)),$$

which leads to the familiar Eq. (6.62) for $\tan(\omega\delta) = \tan(\Delta)$, as expected. After having solved this equation, one can determine the frequency $\omega$ via Eq. (6.63).

Let us now try to apply the same procedure for $\epsilon > 0$ in order to take into account the non-linearity. Looking at the expressions for $A_{j,m}$ and $B_{j,m}$, we see that Eqs. (6.60) and (6.61) not only on coefficients $a_{j,n}$ and $b_{j,n}$ with $n \leq m$ but also on coefficients with $n > m$. This shows that the different modes (i.e. the Fourier terms belonging to different frequencies) are mutually coupled, in contrast to the linear case. One could say, the non-linear term causes a re-distribution over modes of higher frequencies, but also lower ones. In normal fluids, a similar mechanism can lead to turbulent cascades of vortices [?]. For us, the disappointing side effect of this coupling between different frequency modes is that the equations cannot be explicitly solved, at least not exactly. Therefore, we order the coefficients $a_{j,m}$ and $b_{j,m}$ according to size.

The convergence of the infinite series requires that $a_{j,m} \sim \epsilon^m$ and $b_{j,m} \sim \epsilon^m$. We can actually replace $a_{j,m}$ and $b_{j,m}$ by $a_{j,m} \epsilon^m$ and $b_{j,m} \epsilon^m$, which leads to the same results in the limit $\epsilon \rightarrow 1$. Applying this substitution, we get

$$A_{j,m} = \frac{1}{2} \sum_{n=1}^{m-1} (a_{j,n}a_{j,m-n} - b_{j,n}b_{j,m-n}) \epsilon^m + \sum_{n=1}^{\infty} (a_{j,m+n}a_{j,n} + b_{j,m+n}b_{j,n}) \epsilon^{m+2n},$$

$$B_{j,m} = \frac{1}{2} \sum_{n=1}^{m-1} (a_{j,n}b_{j,m-n} + b_{j,n}a_{j,m-n}) \epsilon^m + \sum_{n=1}^{\infty} (b_{j,m+n}a_{j,n} - a_{j,m+n}b_{j,n}) \epsilon^{m+2n}.$$

(6.66)

If we neglect the terms with $\epsilon^{m+2n}$, as for $\epsilon < 1$ they are expected to be smaller than $\epsilon^m$, we get the approximations
\[ A_{j,m} \approx \frac{1}{2} \sum_{n=1}^{m-1} (a_{j,n}a_{j,m-n} - b_{j,n}b_{j,m-n}) \epsilon^m, \]

\[ B_{j,m} \approx \frac{1}{2} \sum_{n=1}^{m-1} (a_{j,n}b_{j,m-n} + b_{j,n}a_{j,m-n}) \epsilon^m. \] (6.67)

These terms depend only on coefficients \( a_{j,n} \) and \( b_{j,n} \) with \( n < m \), which in principle allows on to construct an approximate solution. Specifically, for \( m = 1 \) we get \( A_{j,1} \approx 0 \approx B_{j,1} \) so that the non-linearity does not change the determination equations (6.66), (6.68), and (6.69) for \( \omega \) and \( \delta \), given that we neglect higher-order terms in \( \epsilon \). For \( m = 2 \) we find \( A_{j,2} \approx \frac{1}{2} (a_{j,1}a_{j,1} - b_{j,1}b_{j,1}) \epsilon^2 \) and \( B_{j,2} \approx \frac{1}{2} (a_{j,1}b_{j,1} + b_{j,1}a_{j,1}) \epsilon^2 \). These coefficients determine the amplitude of the mode of frequency \( m\omega = 2\omega \). The related Fourier component corresponds to the correction \( Y_j(t) \) in the perturbative expansion approach of Sec. 6.6. For \( m = 3 \) we get \( A_{j,3} \approx (a_{j,2}a_{j,1} - b_{j,2}b_{j,1}) \epsilon^3 \) and \( B_{j,3} \approx (a_{j,1}b_{j,2} + a_{j,2}b_{j,1}) \epsilon^3 \), which relates to the correction \( Z_j(t) \) in the perturbative expansion.

However, for circular roads it turns out that we have to go beyond the approximation (6.67) and, with this, beyond the perturbation theory pursued in Sec. 6.6, otherwise we cannot see the influence of the non-linearity. The circular closing gives the last vehicle a weak influence on the first one, which causes the additional terms of order \( \epsilon^{m+2n} \) that did not appear in the perturbation approach. A better approximation is given by

\[ A_{j,1} \approx (a_{j,2}a_{j,1} + b_{j,2}b_{j,1}) \epsilon^3 \] (6.68)

and

\[ B_{j,1} \approx (b_{j,2}a_{j,1} - a_{j,2}b_{j,1}) \epsilon^3, \] (6.69)

see Eqs. (6.66) and (6.68). Inserting these expressions into Eqs. (6.66) and (6.68) eventually lead to the improved conditions

\[
\omega = \frac{(\omega_0)^2}{2\gamma} \sin(\Delta) + \frac{\epsilon^2 B}{a_{j,1}/b_{j,1} + b_{j,1}/a_{j,1}} \times \left[ \left( b_{j,2}a_{j,1} - a_{j,2}b_{j,1} \right) \left( \frac{\sin(\Delta)}{a_{j,1}} + \frac{1 - \cos(\Delta)}{b_{j,1}} \right) \right.
\]

\[
+ \left( a_{j,2}a_{j,1} + b_{j,2}b_{j,1} \right) \left( \frac{\sin(\Delta)}{b_{j,1}} - \frac{1 - \cos(\Delta)}{a_{j,1}} \right) \right], \quad (6.70)
\]

which replaces Eq. (6.66). The additional terms proportional to \( \epsilon^2 \) are due to the non-linearity. They change the determination equation for the phase shift \( \delta = \Delta/\omega \), and therefore also the determination equation for the frequency \( \omega \) and speed c. Remarkably, the corrections cause a dependence on the wave amplitudes \( a_{j,m} \) and \( b_{j,m} \). It is obvious that the stability regime changes as well. It also becomes dependent on the wave amplitudes and, thereby, on the perturbation size.
6.8 Critical Densities and Critical Amplitudes

The results of computer simulations of the non-linear optimal velocity model with finite-sized perturbations are similar to the findings for the macroscopic Kerner-Konhäuser model and can be summarized as follows (see Fig. ??): Altogether, there are four critical densities \( \rho_{c_k} \) with \( k \in \{1, 2, 3, 4\} \). Below some density \( \rho_{c_1} \), any kind of disturbance eventually disappears. Between the densities \( \rho_{c_1} \) and \( \rho_{c_2} \), one wide traffic jam builds up, given a large enough perturbation. A series of traffic jams appears in a density range between \( \rho_{c_2} \) and some density \( \rho_{c_3} \). A so-called “anticluster” or “dipole layer” can be triggered [?], if the density \( \varrho \) is between \( \rho_{c_3} \) and \( \rho_{c_4} \), while any disturbance disappears in stable traffic above some density \( \rho_{c_4} \). The critical densities \( \rho_{c_k} \) depend mainly on the choice of the model parameters, in particular the relaxation time and the velocity-distance or velocity-density relation.

Kerner and Konhäuser [?] have shown that, in the density ranges \([\rho_{c_1}, \rho_{c_2}]\) and \([\rho_{c_3}, \rho_{c_4}]\), an existing traffic jam does not trigger any further jams, because traffic flow is not linearly unstable anymore. In these density regimes, they found metastable traffic, which is characterized by a critical amplitude \( \Delta \rho_c(\varrho) \) for the formation of traffic jams. This amplitude is zero for \( \varrho = \rho_{c_2} \) and \( \varrho = \rho_{c_3} \), i.e. at the edge of unstable traffic, while it grows towards the edge of stable traffic and is expected to diverge at \( \varrho = \rho_{c_1} \) and \( \varrho = \rho_{c_4} \). Perturbations with subcritical amplitudes \( \Delta \rho < \Delta \rho_c \) are eventually damped out (analogous to the stable density ranges), while perturbations with supercritical amplitudes \( \Delta \rho > \Delta \rho_c \) grow and form traffic jams (similar to the linearly unstable density ranges). The situation in metastable traffic is, therefore, similar to supersaturated vapor [?], where an overcritical nucleus is required for condensation (“nucleation effect”).

Fig. 6.1. Schematic illustration of the critical amplitude \( \Delta \rho_c(\rho) \) of perturbations of a given form. While perturbations of larger amplitudes grow, perturbations of smaller amplitude eventually fade away. WENJIAN, PLEASE PREPARE THE ILLUSTRATION.

6.9 Further Effects of Non-Linearity

The traffic equations are so highly non-linear that, beyond the determination of the linear instability thresholds, their linear approximation is only of very limited use. When simulating unstable traffic flows in the computer, one can make several interesting observations: For example, the wave number \( \hat{\kappa} \) associated with the largest growth rate \( \lambda_l(\hat{\kappa}) \) as a function of the average density \( \rho_e \) does not at all determine the finally resulting wave length via the formula \( 2\pi/\hat{\kappa} \). Moreover, the forming waves are typically not periodic (see Fig. ??).
Even when starting with a small sinusoidal perturbation, we will not just observe a growth of the wave amplitude, as the linear approximation suggests. Due to the coupling of modes of different frequency (see Sec. ??), the extended perturbation will drastically change its shape, until it eventually becomes localized. Kerner and Konhäuser called this the "local breakdown effect" [?]. The resulting perturbation has a characteristic form which can be approximated by the function

\[\rho(x, t_0) = \rho_e + \Delta \rho \left[ \cosh^{-2} \left( \frac{x - x_0}{w_+} \right) - \frac{w_+}{w_-} \cosh^{-2} \left( \frac{x - x_0 - (w_+ + w_-)}{w_-} \right) \right] \] (6.71)

with suitable parameters \(t_0, x_0, w_+\) and \(w_-\) determining its location and width (see Fig. ??a). When this shape has been reached, the perturbation grows more and more, playing the role of a nucleus for jam formation [?]. This may be compared with the nucleation of water droplets in oversaturated fog [?].

While small-amplitude perturbations flow with the traffic, the propagation speed becomes slower with increasing perturbation amplitude and eventually becomes negative. The final result is a wide traffic jam of characteristic form, which propagates upstream. This is, because vehicles are leaving the standing jam at the front, while new ones are joining the traffic jam at its end. The traffic jam is localized, which is sometimes called the "local cluster effect". Moreover, it is normally surrounded by free traffic flow. Hence, one could say that there is a phase separation (segregation) between free and congested traffic [?]. The result of this noise-induced ordering process [?] reminds of an equilibrium between two different phases (a freely moving, "gaseous" state and a jammed, "condensed" state). In other words, traffic jams absorb as much cars as necessary to have free traffic in the rest of the system. The resulting state of the system is not a partial congestion with all vehicles moving slow, as the velocity-density relation \(V_e(\rho)\) would suggest.

### 6.10 Jam Line and Hysteresis

After some time \(t_0\), a traffic jam on a homogeneous freeway section tends to develop a stationary downstream wave front, which moves upstream with constant velocity \(c = -15 \pm 5\) km/h [?]. This characteristic speed is a result of the self-organized shape of the wave front. The propagation at constant speed allows us to apply the travelling wave approximation

\[\rho(x, t) = \rho(x - c(t - t_0), t_0) \quad \text{and} \quad Q(x, t) = Q(x - c(t - t_0), t_0). \] (6.72)

Inserting this into the continuity equation (??), defining the variable \(x_t = (x - ct)\), and applying the chain rule \(\partial \rho(x_t, t_0)/\partial t = \partial \rho(x_t, t_0)/\partial x \cdot dx_t/\partial t\)
transforms the corresponding partial differential equation into an ordinary differential equation:

\[-c \frac{\partial \rho(x, t_0)}{\partial x} + \frac{\partial Q(x, t_0)}{\partial x} = 0.\] 

(6.73)

This is solved by

\[Q(x, t_0) = Q_0 + c \rho(x, t_0),\] 

(6.74)

where \(Q_0\) is a suitable integration constant \([?]\). Hence, the flow-density relation of a fully developed downstream jam front is a linear curve

\[J(\rho) = Q_0 + c \rho\] 

(6.75)

with a negative slope \(c < 0\), the so-called “jam line”. The negative propagation speed \(c\) reflects that previously stopped vehicles are leaving the jam at its front, which results in an upstream motion. In second-order macroscopic and some other traffic models, the above jam line is a self-organized flow-density relation which can differ significantly from the fundamental diagram \(Q_e(\rho)\) [see, for example, Eqs. (??) and (??)]. Therefore, the fundamental diagram \(Q_e(\rho)\) should be fitted only in the range of stable traffic flow, while congested traffic flow does not necessarily relate to the fundamental diagram.

The intersection point of the jam line \(J(\rho)\) with the free branch of the fundamental diagram \(Q_e(\rho)\) defines the outflow

\[Q_{out} = Q_e(\rho_{out}) = J(\rho_{out})\] 

(6.76)

from congested traffic into free traffic downstream of it. The resulting density of free traffic flow is denoted by \(\rho_{out}\). Moreover, the intersection point \(J(\rho_{jam})\) with the \(\rho\)-axis defines the jam density \(\rho_{max}\). Altogether, we can write

\[J(\rho) = Q_{out} + c(\rho - \rho_{out}) = |c|(\rho_{jam} - \rho).\] 

(6.77)

6.11 Self-Organized Constants of Traffic Jams and Capacity Drop

It turns out that, as a consequence of the self-organized wave fronts resulting from the non-linear traffic dynamics, the parameters \(c, \rho_{out},\) and \(Q_{out}\), but also the density and vehicle speed in traffic jams, are characteristic constants \([?]\), i.e. they are largely independent of the initial conditions and the average density. This is also a consequence of the emergent segregation into areas of congested and free traffic (see Sec. ??). The question is, whether it is possible to calculate the characteristic constants. In fact, for some models this is still feasible (see also Exercise 7...). We will illustrate this for the optimal-velocity model with a simplified optimal velocity function

\[v_o(d_j) = v^0 \Theta(d_j - d_0),\] 

(6.78)
where
\[ \Theta(d_j - d_0) = \begin{cases} 1 & \text{if } d_j - d_0 > 0, \\ 0 & \text{otherwise} \end{cases} \] (6.79)
denotes the Heaviside step function. For this choice, the optimal velocity model can be exactly solved [?]. For our calculations, we assume a number of jammed vehicles with velocity zero and distance \(1/\rho_{\text{jam}}\) jam, where \(\rho_{\text{jam}}\) denotes the jam density. Moreover, if we assume that a car starts to accelerate out of the jam only when its leading car has approximately reached its desired velocity \(v^0\), and starts to decelerate when its predecessor has almost stopped in the traffic jam, we have the acceleration equation \(dv_j/dt \approx (v^0 - v_j(t))/\tau > 0\) for \(d_j > d_0\) and \(dv_j/dt \approx (0 - v_j(t))/\tau < 0\) for \(d \leq d_0\). (The conditions for this approximation are checked in Exercise ...). Consequently, we find
\[ v_j(t) = \begin{cases} v^0 (1 - e^{-(t-t_0)/\tau}) & \text{if } d_j(t) > d_0, \\ v^0 e^{-(t-t_2)/\tau} & \text{otherwise.} \end{cases} \] (6.80)

\(t_0\) is a time point with \(d_j(t_0) = d_0\) and \(dd_j(t_0)/dt > 0\), when the acceleration of vehicle \(j\) starts, while \(t_2^j\) with \(d(t_2) = d_0\) and \(dd_j(t_2)/dt < 0\) is the successive time point, when the deceleration starts.

We may distinguish four different phases:

- Phase 1 is characterized \(dv_{j-1}(t)/dt > 0\) and \(dv_j(t)/dt > 0\), i.e. vehicle \(j\) and its leader \(j-1\) both accelerate. According to the travelling wave concept, the trajectory of vehicle \(j\) is exactly identical with the one of its predecessor, but shifted by some time period \(T\) which corresponds to delay between the acceleration of two successive vehicles from the traffic jam, i.e.
\[ v_j(t) = v_{j-1}(t - T). \] (6.81)

The speed \(c\) of jam resolution is then given by the distance \(-1/\rho_{\text{jam}}\) between two jammed vehicles, divided by this time period \(T\):
\[ c = -\frac{1}{\rho_{\text{jam}} T}. \] (6.82) {ccalc}

The negative sign is a consequence of the fact that vehicle \(j\) stands upstream of vehicle \(j-1\), but accelerates later, so that the downstream front of the traffic jam dissolves opposite to the direction of motion.

Considering the delay \(T\), we get \(dd_j(t)/dt = v_{j-1}(t) - v_j(t) = v_j(t - T) - v_j(t)\), which by integration over time and with \(d_j(t_0) = d_0\) results in
\[ d_j(t) = d_0 + v^0 (1 - e^{-T/\tau}) \tau (1 - e^{-(t-t_0)/\tau}) = d_0 + (1 - e^{-T/\tau}) \tau v_j(t), \] (6.83)

which defines a linear increase of distance with speed (see Fig. ??). Consequently, the desired velocity \(v^0\) is reached at the distance
\[ \frac{1}{\rho_{\text{out}}} = d_0 + (1 - e^{-T/\tau}) \tau v^0, \] (6.84) {quout}
which defines the density $\rho_{\text{out}}$.

- Phase 2 is characterized by $dv_{j-1}(t)/dt < 0$ and $dv_j(t)/dt > 0$, i.e. vehicle $j-1$ already decelerates, while vehicle $j$ still accelerates. This phase is assumed to start at time $t_1 > t_0$, and we have $dd_j(t)/dt = v_0^0 e^{- (t-t_1)/\tau} - v_0^0 (1 - e^{-(t-t_0)/\tau}) < 0$. Therefore, we get

$$d_j(t) = d_j(t_1) - v_0^0 (t-t_1) + \tau v_0^0 \left(1 - e^{-(t-t_1)/\tau}\right). \text{INDICESNICHKTONSTENT??}$$  (6.85)

That is, the distance goes down, while the speed $v_j(t)$ is approximately $v_0^0$.

- Phase 3 starts at some time $t_2 > t_1$ and is characterized by $dv_{j-1}(t)/dt < 0$ and $dv_j(t)/dt < 0$, i.e. vehicle $j-1$ decelerates, and vehicle $j$ does the same with a time delay of $T$. One can say that phase 3 is the inverse process of phase 1, and we get

$$d_j(t) = d_0 - v_0^0 (1 - e^{-T/\tau}) \tau \left(1 - e^{-(t-t_2)/\tau}\right) = d_0 - (1 - e^{-T/\tau}) \tau [v_0^0 - v_j(t)].$$  (6.86)

Accordingly, the distance is monotonously decreasing with time. The minimum distance is reached for the jam density $\rho_{\text{jam}}$, which is defined by

$$\frac{1}{\rho_{\text{jam}}} = d_0 - (1 - e^{-T/\tau}) \tau v_0^0.$$  (6.87)

Together with Eq. (6.84) we find

$$\frac{1}{\rho_{\text{out}}} - \frac{1}{\rho_{\text{jam}}} = 2\tau v_0^0 (1 - e^{-T/\tau}).$$  (6.88)

- Phase 4 starts at time $t_3$ and is characterized by $dv_{j-1}(t)/dt > 0$ and $dv_j(t)/dt < 0$, i.e. vehicle $j$ still decelerates, while its leader $j-1$ already accelerates. Phase 4 is the inverse process of phase 2. We find

$$d_j(t) = \frac{1}{\rho_{\text{jam}}} + v_0^0 (t-t_3) - \tau v_0^0 (1 - e^{-(t-t_3)/\tau}).$$  (6.89)

That is, the distance grows linearly in time, while the velocity of vehicle $j$ is approximately zero. The distance $d_j(t)$ becomes $d_0$ at time $t_4$. The difference $t_4 - t_3$ determines the time delay $T$ between two successive acceleration maneuvers of cars leaving the traffic jam. With definition (6.87) we find

$$d_0 = \frac{1}{\rho_{\text{jam}}} + v_0^0 T - \tau v_0^0 \left(1 - e^{-T/\tau}\right) = d_0 + v_0^0 T - 2\tau v_0^0 \left(1 - e^{-T/\tau}\right)$$  (6.90)

or an implicit relationship for the time shift

$$T = 2\tau \left(1 - e^{-T/\tau}\right).$$  (6.91)
According to Eqs. (6.87), (6.84), and (6.91), it was possible to express the characteristic constants $T$, $c$, $\rho_{\text{jam}}$, and $\rho_{\text{out}}$ through the model parameters $d_0$, $v^0$, and $\tau$ only, without any dependence on the initial condition. Note that all the characteristic parameters are no model parameters. They are rather a result of the self-organization of characteristic jam fronts (see Fig. ??).

It is also worth stating that the outflow

$$Q_{\text{out}} = \rho_{\text{out}} v^0 = \frac{1}{d_0/v^0 + T/2} = \frac{2}{T} \left( 1 - \frac{1}{1 + 2d_0/(v^0T)} \right)$$

(6.92)

is smaller than the maximum flow

$$Q_{\text{max}} = \frac{v^0}{d_0} = \frac{2}{T},$$

(6.93)

that is, there is a density region in which two different flows are possible. The actually assumed density value depends on the history. This is called a hysteresis effect. WENJIAN, PLEASE PRODUCE ILLUSTRATIONS FOR THIS Once a traffic jam forms, there is an effective capacity drop of size

$$\Delta Q_{\text{drop}} = Q_{\text{max}} - Q_{\text{out}} = \frac{2}{T} \frac{2d_0/(v^0T)}{1 + 2d_0/(v^0T)}.$$ 

(6.94)

This implies a reduction of freeway capacity by traffic breakdowns, i.e. particularly during the rush hours, when more capacity would be urgently needed. A successful traffic control strategy is, therefore, to avoid or delay traffic breakdowns by suppressing perturbations, i.e. by homogenizing the flow. This can be done by adaptive speed limits, traffic-responsive ramp metering, dynamic re-routing, or cooperative driving based on driver assistance systems [?].

It is remarkable that the fundamental diagram of the above specification of the optimal velocity model assumes a vanishing flow in the congested area. However, the effectively resulting flow-density relation for fully developed traffic jams is given by the jam line

$$J(\rho) = \frac{1}{T} \left( 1 - \frac{\rho}{\rho_{\text{max}}} \right),$$

(6.95)

which follows from Eqs. (??) and (6.82). In other words: While the velocity-density relation according to the optimal velocity function $V_o(d)$ is given by

$$V_o(\rho) = v_o(1/\rho) = \left\{ \begin{array}{ll} v^0 & \text{if } \rho < 1/d_0, \\ 0 & \text{otherwise,} \end{array} \right.$$ 

(6.96)

the effectively resulting speed-density relationship, considering the self-organization of unstable traffic flow is given by the considerably different function

$$V_{\text{eff}}(\rho) = \frac{J(\rho)}{\rho} = \left\{ \begin{array}{ll} \frac{v^0}{T} \left( 1 - \frac{1}{\rho_{\text{jam}}} \right) & \text{if } \rho < 1/d_0, \\ \frac{1}{T} \left( 1 - \frac{1}{\rho_{\text{jam}}} \right) & \text{otherwise.} \end{array} \right.$$ 

(6.97)
Therefore, it does not make much sense to measure the fundamental diagram in the unstable density regime.

Summarizing the above results, one finds characteristic traffic constants, if the jam density \( \rho_{\text{jam}} \) developing at the upstream end of traffic jams is independent of the surrounding traffic. In contrast to \( \rho_{\text{jam}} \), the time interval \( T \) between accelerating vehicles is determined by the dynamics at the downstream jam front. It results from the fact that the initial conditions of accelerating vehicles at jam fronts are more or less identical: The cars have equal distances \( d_{\text{jam}} \approx 1/\rho_{\text{jam}} \) to the respective leading vehicle and start with the same velocity \( v \approx 0 \).

### 6.12 Calculation of Critical Densities and Critical Amplitudes

Encouraged by our success in calculating the characteristic constants for the simplest possible implementation of the optimal velocity model, let us now try to derive the critical densities and critical amplitudes for this model as well. Let us again study a circular one-lane system of length \( L \) with \( N \) vehicles and an average vehicle density \( \rho = N/L \). In order to avoid finite size effects, we will assume a large system with many vehicles. According to Eq. (??), if all vehicles have a distance greater than \( d_0 \) to their leader, we expect free traffic at speed \( v_0 \), while we expect congested, standing traffic, if all vehicles have a distance smaller than \( d_0 \). Therefore, an interesting dynamics will occur only of some
vehicles have distances larger than $d_0$, while others have distances smaller than $d_0$. In the following, we will focus on this case. One first observation is that linearly unstable traffic should be possible only for $\rho = N/L = 1/d_0$, so that for the particular specification (?) of the optimal velocity function $v_o(d)$ we have the special case
\[
\rho_{c2} = \rho_{c3} = \frac{1}{d_0}.
\]  
(6.98)
But what is the value of the critical density $\rho_{c1}$ at which non-linear instability starts to be possible, and what is the critical density $\rho_{c4}$, beyond which even arbitrarily large perturbations will fade away? And can we determine the critical amplitudes $\Delta \rho_{cr}$ (see Sec. ...)?

Let us in the following focus on a special kind of perturbation: We will assume that $(N-1)$ vehicles have an identical distance $d_0$ to their respective predecessor, while the distance $d_1(0)$ of the first vehicle to its predecessor is $d_1(0) = L - (N-1)d_0$. Accordingly, we define the perturbation size as
\[
\Delta \rho = \left| \frac{1}{d_0} - \frac{1}{d_1(0)} \right| = \left| \frac{1}{d_0} - \frac{1}{L - (N-1)d_0} \right| = \left| \frac{L/d_0 - N}{L - (N-1)d_0} \right| \geq 0.
\]  
(6.99)
In order to avoid finite-size effects, we will assume a very large system of length $L = N/\rho$ with $N \to \infty$ vehicles. Then, we have
\[
d_0 = \frac{L - d_1(0)}{N-1} \to \frac{L}{N} = \frac{1}{\rho}.
\]  
(6.100)
As we know that the propagation of perturbations requires at least one vehicle distance (either $d_0$ or $d_1(0)$) to be above and another one below $d_0$, we can estimate lower bounds for the critical perturbation amplitudes as follows:
\[
\Delta \rho_{cr} \geq \left| \frac{1}{d_0} - \frac{1}{\rho} \right| = \left| \frac{1}{d_0} - \frac{1}{d_0} \rho d_0 \right| = \left| \frac{1 - \rho d_0}{d_0} \right| = \left| \frac{1 - \rho d_0}{d_0} \right| \geq 0.
\]  
(6.101)
We can see that the critical amplitude can become zero only for $\rho = 1/d_0 = \rho_{c2} = \rho_{c3}$. However, formula (6.101) does not indicate any divergence of the critical amplitudes, which would be the precondition to define critical densities $\rho_{c1}$ and/or $\rho_{c4}$. Therefore, we need a better estimation of the critical amplitudes. In our further analysis, we will treat the case of large densities separately from the case of small densities.

In the case of large densities $\rho \geq \rho_{c3} = 1/d_0$, an initial perturbation may fade away if the maximum vehicle distance does not allow to reach a large enough speed in the acceleration process in order to reach a vehicle distance equal to or smaller than $d_0$ after its successive braking maneuver, if the perturbation is not large enough. While the speed in the vehicle queue is approximately zero (despite for, maybe, the last few vehicles in the queue which may still decelerate), the maximum speed reached after an acceleration time of $t_0$ is
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\[ v_{(0)} = v^0 \left( 1 - e^{-t(0)/\tau} \right), \quad (6.102) \]

as Eq. (???)) implies \( v_1(t) = v^0(1 - e^{-t/\tau}) \). If the first vehicle’s position at time \( t = 0 \) is set to \( x_1(0) = 0 \), the temporal evolution of its location is therefore given by

\[ x_1(t) = \int_0^t dt' v_1(t') = v^0 \left( t - \tau e^{-t/\tau} \right), \quad (6.103) \]

At time \( t(0) \) when the first vehicle starts to decelerate, the distance to the last vehicle in the queue must be \( d_0 \). As its location is at \( x = L - (N - 1)d_0 \), this implies

\[ L - (N - 1)d_0 - x_1(t(0)) = d_0 \quad (6.104) \]

or

\[ L - (N - 1)d_0 - d_0 = v^0 \left( t(0) - \tau e^{-t(0)/\tau} \right), \quad (6.105) \]

which is an implicit equation determining the acceleration time period \( t(0) \). At the end of its deceleration process, the previously first vehicle of the queue, which has then joined the end of the queue, will have a distance \( d_{(1)} \) which depends on the maximum speed \( v(0) \). As the deceleration process according to Eq. (???)) is follows the exponential velocity decay

\[ v(t) = v(0)e^{-[t-t(0)]/\tau}, \quad (6.106) \]

the resulting minimum distance can be determined as

\[ d_{(1)} = d_0 - \int_{t(0)}^\infty dt \, v(0)e^{-[t-t(0)]/\tau} = d_0 - v(0)\tau. \quad (6.107) \]

The other vehicles in the queue are expected to have the same distance after one cycle of acceleration and deceleration. That is, while their distance was \( d_0 \) in the beginning, the distance will be \( d_{(1)} \) after one cycle, \( d_{(2)} \) after two cycles, and so one. The iterative equations to determine the decisive quantities are listed below and completely analogous to the equations derived above:

\[ d_{(n+1)} = d_0 - v(n)\tau = d_0 - v^0\tau \left( 1 - e^{-t(n)/\tau} \right), \quad (6.108) \]

where \( t(n) \) is determined by the implicit equation

\[ L - (N - 1)d(n) - d_0 = v^0 \left( t(n) - \tau e^{-t(n)/\tau} \right), \quad (6.109) \]

which has a unique solution. Considering Eqs. (6.108) and (6.109), the first one of these two equations may also be replaced by the simpler relationship

\[ d_{(n+1)} = L - (N - 1)d(n) - v^0 t(n)(d(n)), \quad (6.110) \]
where \( t_{(n)}(d_{(n)}) \) is obtained from Eq. (6.109). So, the interesting question is whether the series \( d_{(n)} \) converges and, if yes, to what value. If the values \( d_{(n)} \) stay the same for different values of \( n \), the initial perturbation is stable over time. If the distances go down, then the perturbation grows (which is expected to happen for a large enough initial perturbation, see Exercise ....). However, if the values of \( d_{(n)} \) grow with \( n \), the initial perturbation is expected to fade away (which is expected to happen, when it is too small). Therefore, the critical amplitude is given by the condition \( d_{(n+1)} = d_{(n)} \), which together with Eq. (6.110) implies \( d_{(n)} = L - (N - 1)d_{(n)} - \frac{v^0}{\tau}t_{(n)} \) or

\[
\tau \left( 1 - e^{-t_{(n)}/\tau} \right) = d_{(n)} = L - \frac{1}{\rho} = \frac{1}{\rho} + \frac{1}{\rho} \frac{d_0 - L}{v^0} = 0. 
\]  

This corresponds to the homogeneous solution, in which the perturbation has disappeared. A rough approximation of the second solution is obtained by the second-order Taylor expansion \( e^{-t/\tau} \approx 1 - t/\tau + t^2/(2\tau^2) \), for which Eq. (6.109) becomes

\[
\frac{N(t_{(n)})^2}{2\tau} - (N - 1)t_{(n)} + \frac{Nd_0 - L}{v^0} = 0. 
\]  

The two solutions of this polynomial are given by

\[
t_{(n)} = \frac{N}{\tau} \left( 1 \pm \frac{\sqrt{(N - 1)^2 - 4Nd_0 - L}}{2\tau} \right) \to \tau \left( 1 \pm \frac{1}{\tau \sqrt{1 - \frac{2(d_0 - L)}{\rho}}} \right). 
\]  

The limiting case on the right-hand side is for \( N \to \infty \). Considering the first-order Taylor expansion \( \sqrt{1 - \frac{1}{x}} \approx 1 - \frac{1}{2x} \), the minus sign leads again to the previous solution \( t_{(n)} \approx (d_0 - 1/\rho)/v^0 \). The plus sign, however, gives us the additional solution...
which may be compared with Eq. (??) \( T = 2\tau(1 - e^{-T/\tau}) \). Inserting the second solution for \( t(n) \) into Eq. (6.111) finally gives the stationary value of \( d(n) \), which is given by

\[
d(n) = \frac{L}{N} - \frac{v^0 t(n)}{N} = \frac{1}{\bar{\rho}} - \frac{2\tau v^0}{N} + \frac{d_0 - 1}{\bar{\rho} N} - \frac{1}{\bar{\rho}},
\]

and defines the critical perturbation

\[
\Delta \rho_{cr} = \frac{1}{d(n)} - \frac{1}{L - (N-1)d(n)} \approx \bar{\rho} - \left[ \frac{1}{\bar{\rho}} + \frac{(N-1)}{N} \left( 2v^0 \tau - d_0 + \frac{1}{\bar{\rho}} \right) \right]^{-1}
\]

\[
\approx \bar{\rho} - \frac{1}{2(2v^0 \tau - d_0)\bar{\rho}}
\]

\[\text{secor}\]

see Eq. (??). The limit on the right-hand sides hold for \( N \to \infty \). It is obvious that the critical amplitude depends on the average density \( \bar{\rho} \). Within the limitations of our above approximation, as expected we have \( \Delta \rho_{cr} \to 0 \) for

\[
\bar{\rho} = \frac{1}{d_0 - 2v^0 \tau} \approx \rho_{c3},
\]

where we assume small values of \( \tau \). Moreover, the critical amplitude \( \Delta \rho_{cr}(\bar{\rho}) \) diverges for

\[
\bar{\rho} \to \frac{2}{d_0 - 2v^0 \tau} = \rho_{c4} \approx 2\rho_{c3},
\]

which defines the critical threshold \( \rho_{c4} \).

In the case of small densities \( \bar{\rho} \leq \rho_{c2} = 1/d_0 \), the survival of a perturbation requires that there is at least one vehicle with a distance \( d_{min} = L-(N-1)d(0) \) smaller than \( d_0 \) for a long enough time to force the successive vehicle to brake. It may in fact happen that this vehicle already starts to accelerate before the follower reaches it, which causes the perturbation to fade away. Therefore, we must determine how long it takes until the vehicle with distance \( d_{min} \) starts to accelerate, assuming for simplicity that its velocity is approximately zero before. Considering Eq. (??), this time period \( T \) is given by the implicit equation

\[
d_{\text{min}} + v^0 \left( T - \tau + \tau e^{-T/\tau} \right) = d_0,
\]

as the vehicle starts to accelerated when its leader has reached a distance \( d_0 \). Note that this relationship generalizes Eq. (??) to values \( d_{\text{min}} \neq \rho_{\text{jam}} \). In the crudest approximation we can make, we require that the time period \( T \) is not smaller than the time period \( t_* \) of the first successor, before he or she starts decelerating. This time period is given by its distance \( d(0) \approx 1/\bar{\rho} \), see Eq. (??). It is determined from the point in time when the measurement of the time period \( T \) is started (without loss of generality at time \( t = 0 \)). The time period
6.12 Calculation of Critical Densities and Critical Amplitudes

$t_*$ is given by the equation $d_{(0)} - v^0 t_* = d_0$, because the deceleration process starts when the distance has shrunk to $d_0$. As a consequence,

$$t_* = \frac{d_{(0)} - d_0}{v^0}. \quad (6.123)$$

This shall be equal to $T$. In second order Taylor approximation of Eq. (6.122) we obtain the polynomial

$$d_{\min} + v^0 \frac{T^2}{2\tau} = d_0, \quad (6.124)$$

which has the positive solution

$$T \approx \sqrt{\frac{2\tau (d_0 - d_{\min})}{v^0}}. \quad (6.125)$$

Therefore, the requirement $t_* \leq T$ for the survival of the perturbation implies $(t_*)^2 \leq T^2$ and finally

$$d_{\min} \leq d_0 - \frac{(1/\rho - d_0)^2}{2\tau v^0}. \quad (6.126)$$

The critical perturbation amplitude is obtained for the equality sign. Because of $d_{\min} = L - (N - 1)d_{(0)}$, it reads

$$\Delta \rho_{\text{cr}} = \left| \frac{1}{1/\rho} - \frac{1}{d_{\min}} \right| = \bar{\rho} - \frac{1}{d_0 - \frac{(1/\rho - d_0)^2}{2\tau v^0}}. \quad (6.127)$$

Note that the critical amplitude becomes zero for $\bar{\rho} = 1/d_0 = \rho_{c2}$, as expected. Moreover, it diverges, when the denominator becomes zero, which implies $(1/\bar{\rho} - d_0)^2 = 2\tau v^0 d_0$ or

$$\bar{\rho} = \frac{1}{d_0 + \sqrt{2\tau v^0 d_0}} = \rho_{c1}. \quad (6.128)$$

This density at which the critical amplitude diverges finally defines the critical density $\rho_{c1}$ we were looking for. Note, however, that this calculation served to illustrate the idea only of how to determine the critical amplitude and density. It assumed many simplifications. For example, a more accurate calculation would have to take into account that the minimum $v_{(n)}$ vehicle velocity in the small-density case may be finite and that the speed $v_{(n+1)}$ of the successor in our above consideration must reach this velocity exactly in the case of stable perturbations before accelerating again. Generally, however, the values of $v_{(n)}$ constitute an infinite series?????????, which will normally converge against a certain value. If it converges against $v^0$, the perturbation will fade away, otherwise it will survive.
6.13 Instability Diagram

We can now summarize the findings of sections ??, ??, and ?? in an instability diagram (see Fig. 6.3). Assuming a model with a fundamental diagram $Q_e(\rho)$, the low densities with $dQ_e/d\rho > 0$ define the area of free traffic and the densities with $dQ_e(\rho)/d\rho < 0$ the area of congested traffic. Traffic flow at very low and very high densities is usually linearly stable. However, if the linear instability conditions of Chap. ?? are fulfilled, we find that small perturbations can grow. This concerns the medium density regime between two critical densities $\rho_{c2}$ and $\rho_{c3}$, where the average vehicle speed changes strongly with the distance (or density). For densities slightly below, namely between some critical threshold $\rho_{c1}$ and $\rho_{c2}$, traffic flow is metastable, i.e. perturbations larger than some critical amplitude $\Delta \rho_c(\rho)$ grow, otherwise they fade away (see Fig. 6.1). This amplitude depends on the shape of the initial perturbation. It is zero at density $\rho_{c2}$ and grows towards $\rho_{c1}$. The average wave length of the resulting traffic jams diverges \[?\].

The metastable regime right of $\rho_{c3}$ has similar features as the one left of $\rho_{c2}$. Between $\rho_{c3}$ and another critical threshold $\rho_{c4}$, traffic is metastable, while for higher densities, no perturbation can trigger traffic jams (characterized by standing vehicles). The critical amplitude $\Delta \rho_c(\rho)$ is zero at $\rho_{c3}$ and large at $\rho_{c4}$. Finally, the density $\rho_{out}$ belonging to the outflow $Q_{out}$ from congested traffic typically lies typically between the densities $\rho_{c1}$ and $\rho_{c2}$, i.e. it is metastable. While the above picture summarizes the most important findings to understand congested traffic patterns, one may distinguish further details \[?\], which complicate matters.

Interestingly, there are also traffic models without the phenomenon of metastability. Investigations by Krauß \[?\] for a Gipps-like family of traffic models \[?\] suggest the following general conclusions (see Fig. 6.4):

(i) Models with a high ratio between maximum acceleration $a$ and maximum deceleration $b$ never show any structure formation, since traffic flow is always stable. This behavior is also found for the optimal velocity model with small relaxation times $\tau > 0$ with $1/(2\tau) \geq \max_d dv_o(d)/dd$.

(ii) Models with a relatively high maximum deceleration $b$ display a jamming transition, which is not hysteretic. As a consequence, there are no metastable high-flow states and the outflow from jams agrees with the maximum flow.

(iii) A hysteretic jamming transition with metastable high-flow states and a characteristic, reduced outflow $Q_{out}$ from traffic jams (see Sec. ??) is found for relatively small accelerations and medium decelerations. This corresponds to the behavior that we have discussed in the previous sections.

The existence of high-flow states requires the flow $Q_{out} = Q_e(\rho_{out})$ to be smaller than the flow $Q_{c2} = Q_e(\rho_{c2})$ at which traffic becomes linearly unstable, so that there is a density range of metastable traffic. By variation of model parameters, it is also possible to have the case $Q_{c2} < Q_{out}$, where
Fig. 6.3. Schematic illustration of velocity $V$ and flow $Q$ as a function of the vehicle density $\rho$. Grey regions indicate density ranges of metastable traffic flow (cf. text), while $Q_{c_k} = Q(\rho_{c_k})$ denote the flows belonging to the instability thresholds $\rho_{c_k}$. The dashed linear relationship corresponds to the jam line $J(\rho)$. Its intersection point with the free branch of the fundamental diagram defines the values the characteristic outflow $Q_{\text{out}}$ and the density $\rho_{\text{out}}$.

traffic breaks down before it can reach high-flow states. This corresponds to case (ii) of the above classification by Krauß [?, ?], where traffic flow never exceeds the jam line $J(\rho)$ (see Fig. 6.4).

6.14 Exercises

1. Determine the first three derivatives of the functions $\sin(x)$, $\cos(x)$, $\exp(x)$ and $\sqrt{b + x}$ in the point $x = 0$ and derive the 3rd order Taylor expansions from this. Do you have any idea how the infinite series expansion could look like? Derive the Euler formula $e^{i\varphi} = \cos(\varphi) + i\sin(\varphi)$ from the infinite series expansions of $\sin(x)$, $\cos(x)$, and $\exp(x)$. Check that $\cos(-\varphi) = \cos(\varphi)$, but $\sin(-\varphi) = -\sin(\varphi)$. Finally, show
Fig. 6.4. (a) Schematic phase diagram of model classes as a function of the acceleration strength $a$ and the deceleration strength $b$. (b) Models belonging to phase (i) have no critical density $\rho_{cr} = \rho_{c2}$ and display stable traffic in accordance with the fundamental diagram (thick dashed and solid lines), while models belonging to phase (ii) show a non-hysteretic jamming transition with a resulting flow-density relation represented by solid lines. (c) A hysteretic phase transition with metastable high-flow states is found for models belonging to phase (iii). The model behavior is determined by the relative magnitude of the outflow $Q_{out}$ from congested traffic compared to the flow $Q_{cr} = Q_e(\rho_{c2})$ at which traffic becomes linearly unstable with respect to small perturbations. (After Krauß, 1998a).
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$$\sum_{n=0}^{m} x^n = \frac{1 - x^{m+1}}{1 - x} \quad (6.129)$$

and derive the infinite series expansion of $1/(1 - x)$ from it.

2. Express $\sin(\varphi)$ and $\cos(\varphi)$ through $\tan(\varphi) = \sin(\varphi)/\cos(\varphi)$, considering $\sin^2(\varphi) + \cos^2(\varphi) = 1$.

3. Derive all sum and trigonometric difference rules [like $\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$] by evaluating $\cos(\alpha \pm \beta\alpha) + \sin(\alpha \pm \beta)$. For this, start with the representation (5.12) of complex numbers, consider $i^2 = -1$ when performing the multiplication of two complex numbers, and take into account the independence of the real and imaginary part of complex numbers. Derive the result of $\sin^2(\alpha) + \cos^2(\alpha)$.

4. Multiply two infinite series $\sum_{m=1}^{\infty} a_m$ and $\sum_{n=1}^{\infty} b_n$ with each other. Assuming the required convergence properties, show that the product can be represented in the following ways:

$$\left(\sum_{m=1}^{\infty} a_m\right)\cdot \left(\sum_{n=1}^{\infty} b_n\right) = \sum_{k=1}^{\infty} \sum_{l=1}^{k-1} a_l b_{k-l} \quad (6.130)$$

and

$$\left(\sum_{m=1}^{\infty} a_m\right)\cdot \left(\sum_{n=1}^{\infty} b_n\right) = \sum_{k=1}^{\infty} a_k b_k + \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_{l+k} b_l + \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} a_l b_{l+k} \quad (6.131)$$

5. Perform a linear and non-linear stability analysis for two identical vehicles on a circular road, moving according to the optimal velocity model (??). For this, show $\delta d_{j-1}(t) = -\delta d_j(t)$ and derive that the real part of the eigenvalues is always negative.
7 Spatial Inhomogeneity

In the last chapter, we have studied the dynamics of transport processes in time and homogeneous space. We will now address the question what happens in cases of spatial inhomogeneities, which imply a variation of capacity in space.

7.1 Source and Sink Terms (In- and Outflows)

Let us now study the spatio-temporal traffic dynamics expected along a freeway section with a single bottleneck such as an on-ramp. Later on, we will discuss scenarios with several bottlenecks as well. Compared to the previous sections, we expect a richer dynamics due to the fact that we have a dynamic velocity equation, which implies unstable and metastable traffic flow regimes.

When the number of lanes is variable, it can be useful to focus on the vehicle density \( \rho'(x,t) \) and flow \( Q'(x,t) \) per lane. From Exercise 2.5 we know that a variation of the width \( I(x) \) along the road causes an additional term

\[
\nu_+(x,t) = -Q'(x,t) \frac{dI(x)}{dx}
\]  

compared to the continuity equation (…), see Eq. (15.12). This shows how on-ramp flows may be treated. Let us imagine an on-ramp entering an \( I \)-lane freeway, centered around the location \( x_{rmp} \). Then, the on-ramp flow \( Q_{on} \) must be substituted for \( Q'(x,t)/I(x,t) \), as in the formula above, this represents the flow that is additionally squeezing into the remaining number of lanes. Furthermore, if the on-ramp flow \( Q_{on}(t) \) is equally distributed over an effectively used ramp length of \( L_{eff} \), we have \( dI(x)/dx = 1/L_{eff} \). Altogether, this implies an inflow of

\[
\nu_+(x,t) = \begin{cases} 
Q_{on}(t) & \text{for } x_{rmp} - L_{eff}/2 < x < x_{rmp} + L_{eff}/2, \\
I_{fr}L_{eff} & \text{otherwise,}
\end{cases}
\]  

(7.2)
where \( I_{fr} = I(x_{rmp} \pm L_{eff}/2) \) denotes the number of freeway lanes upstream and downstream of the ramp. This must be inserted into the continuity equation. Dropping the prime (') for simplicity of notation, we have

\[
\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial}{\partial x} \left[ \rho(x,t)V(x,t) \right] = \nu_+ (x,t) - \nu_- (x,t). \tag{7.3}
\]

The sink term

\[
-\nu_-(x,t) = \begin{cases} 
\frac{Q_{off}(t)}{I_{fr}L_{eff}} & \text{for } x_{rmp} - L_{eff}/2 < x < x_{rmp} + L_{eff}/2, \\
0 & \text{otherwise}
\end{cases} \tag{7.4}
\]

has been subtracted in order to allow one to treat off-ramp flows \( Q_{off}(t) > 0 \) as well.

Obviously, an on-ramp flow \( Q_{on}(t) \) causes a bottleneck, as it consumes some of the capacity of the freeway. The bottleneck strength is given by the entering ramp flow, divided by the number \( I_{fr} \) of freeway lanes:

\[
\Delta Q(t) = \frac{Q_{on}(t)}{I_{fr}}. \tag{7.5}
\]

### 7.2 Congested Traffic States

When simulating traffic flow with the macroscopic Kerner-Konhäuser model, but the parameter set chosen by Lee et al. [26], we find free traffic flow and different kinds of congestion patterns, when the ramp flow \( Q_{on} \) and the upstream arrival flow \( A = Q_{up} \) on the freeway are varied. The diversity of traffic patterns is

1. due to the possibility to have either locally constraint or spatially extended (e.g. growing) congestion and
2. due to the possibility to have stable, unstable or free traffic flows.

Typical representatives of congested traffic patterns are shown in Fig. 7.1, as obtained by computer simulations with the gas-kinetic based traffic model [?].

One can distinguish the different traffic states (i.e. congestion patterns) by analyzing the temporal and spatial dependence of the average velocity \( V(x,t) \). If \( V(x,t) \) stays above a certain threshold \( V_{crit} \), where \( x \) is varied within a homogeneous freeway section upstream of a bottleneck, we call the traffic state free traffic (FT), otherwise congested traffic. If these speeds fall below \( V_{crit} \) only over a short freeway subsection, and the length of this section is approximately stable or stabilizes over time, we talk about localized clusters (LC), otherwise of spatially extended congestion states.

\(^{1}\) A typical threshold for German freeways would be \( V_{crit} \approx 80 \text{ km/h} \).
According to our simulations, there are two forms of localized clusters: Pinned localized clusters (PLC) stay at a fixed location over a longer period of time, while moving localized clusters (MLC) propagate upstream with the characteristic speed $c$. Stop-and-go waves (SGW) may be interpreted as a sequence of several moving localized clusters. Alternatively, they may be viewed as a special case of oscillating congested traffic (OCT), but with free traffic flows of about $Q_{\text{out}}$ between the upstream propagating jams. Generally, however, OCT states are just characterized by oscillating speeds in the congested range, i.e. unstable traffic flows. If the speeds are congested over a spatially extended area, but not oscillating, we call this homogeneous congested traffic (HCT). This is typically related with low velocities and either assumes a model for identical driver-vehicle units or a spatio-temporal smoothing over reasonable temporal and/or spatial intervals (see Sec. ??).

In summary, besides free traffic, traffic models such as the Kerner-Konhäuser model predict five different, spatio-temporal patterns of congested traffic states at a simple on-ramp bottleneck: PLC, MLC, SGW, OCT, and HCT. Similar traffic states have been identified for flow-conserving bottlenecks in a car-following model [?, ?]. Note that, in empirical data, the SGW state may be hard to distinguish exactly from OCT states, due to their similar appearance.

### 7.3 Phase Diagram of Traffic States

It turns out that the possible traffic patterns are mainly determined by the instability diagram (see Sec. ??), no matter if the model is macroscopic or microscopic. Therefore, the preconditions for the possible occurrence of the different traffic states can be illustrated by a phase diagram (see Fig. 7.2). Each area of the phase diagram represents the combinations of upstream freeway flows $A = Q_{\text{up}}$ and bottleneck strengths $\Delta Q = Q_{\text{on}}/I_{\text{fr}}$, for which certain kinds of traffic states can exist. However, the phase diagram can not only be numerically determined. It turns out that the borderlines between different areas (the so-called phase boundaries) can be theoretically understood, based on the flows

$$Q_{\text{ck}} = Q_c(\rho_{\text{ck}})$$

at the instability thresholds $\rho_{\text{ck}}$, the maximum flow capacity $Q_{\text{max}}$ under free flow conditions, and the dynamic flow capacity, i.e. the characteristic outflow $Q_{\text{out}}$ from congested traffic [?] (see Fig. ??). Note, however, that the exact shape and location of the separation lines between different kinds of traffic states depend on the traffic model and selected parameter values, as the characteristic constants of traffic flow and the critical densities vary with the model parameters. Therefore, the separation lines (“phase boundaries”) are also expected to depend on the behavior of drivers, which may be adapted to the traffic or weather conditions (or to the variable speed limits displayed along many freeways).
Fig. 7.1. Simulation of traffic on a freeway with an on-ramp at location \( x = 10 \) km, assuming an initial disturbance travelling upstream and using the non-local, gas-kinetic-based traffic (GKT) model. Depending on the respective traffic flows on the ramp and on the freeway, five different kinds of congested traffic states emerge: a moving localized cluster (MLC), a pinned localized cluster (PLC), stop-and-go waves (SGW), oscillating congested traffic (OCT), or homogeneous congested traffic (HCT). During the first minutes of the simulation, the flows on the freeway and the on-ramp were increased from low values to their final values.

Moreover, note that the density \( \rho_{c2} \) may be smaller or larger than the density \( \rho_{\text{max}} \) belonging to the maximum flow \( Q_{\text{max}} \). Previous computer simulations and phase diagrams mostly assumed parameters with \( \rho_{c2} < \rho_{\text{max}} \). In the following, we will focus on the case \( \rho_{c2} > \rho_{\text{max}} \), as it can explain an additional traffic pattern, the widening synchronized traffic \([?]\). Simpler cases will be addressed in Exercise 7....

### 7.3.1 Transition to Congested Traffic

In the following, we restrict our considerations to situations with one bottleneck only, namely a single on-ramp. Combinations of off- and on-ramps are not covered by this section. They will be treated later on (see Sec. ??).

For matters of illustration, we assume a typical rush hour scenario, in which the total traffic volume

\[
Q_{\text{tot}}(t) = A(t) + \Delta Q(t),
\]

(7.7)
i.e. the sum of the upstream arrival flow \( A \) and the on-ramp flow \( \Delta Q \) per freeway lane is increasing with time \( t \). As long as traffic flows freely, we have
7.3 Phase Diagram of Traffic States

Fig. 7.2. Schematic representation of phase diagrams of traffic states assuming (a) negligible and (b) large perturbations of the traffic flow. The different areas indicate, for which combinations of the upstream freeway flow \( A = Q_{up} \) and the bottleneck strength \( \Delta Q \) certain traffic states (see Fig. 7.1) are predicted to exist. Superposing both diagrams shows that many areas are multistable. That is, depending on the initial and boundary conditions, one may find one out of several possible states. Note that the value of \( Q_{max} \) depends on the speed limit \( V^0 \). SGW states are a special case of OCT states, in which the traffic between the congested areas flows freely. The existence of PLC and MLC states for a certain range of flows indicates that their actual outflow can assume a range of different values between \( Q_{c1} \) and \( Q_{c2} \). PLEASE NOTE: THE LETTER A IN THIS PRELIMINARY GRAPH HAS NOTHING TO DO WITH \( A = Q_{up} \) IN THE TEXT
the flow $A(t)$ upstream of the bottleneck, and downstream of it we have the flow $Q_{\text{tot}}(t)$.

When the total flow $Q_{\text{tot}}$ exceeds the critical density $\rho_{c1}$, it enters the metastable density regime. That is, large enough perturbations may potentially grow and cause a breakdown of the traffic flow. However, the total traffic volume will often rise so quickly that it eventually exceeds the maximum freeway capacity

$$Q_{\text{max}} = \max_\rho Q_e(\rho).$$

(7.8)

When this happens, a queue of vehicles will form upstream of the bottleneck. The flow downstream of the bottleneck is limited to $Q_{\text{max}}$, and as the on-ramp flow takes away an amount $\Delta Q$ of the maximum capacity $Q_{\text{max}}$, the flow upstream of the bottleneck is given by

$$Q_{\text{bot}} = Q_{\text{max}} - \Delta Q.$$  

(7.9)

According to Eq. (7.9), the upstream front of the forming vehicle queue is expected to propagate upstream at the speed

$$C_1 = \frac{A - Q_{\text{bot}}}{\rho_{fr}(A) - \rho_{cg}(Q_{\text{bot}})},$$

(7.10)

where $\rho_{fr}(Q)$ is the density with the free flow $Q_e(\rho_{fr}(Q)) = Q$ and $\rho_{cg}(Q)$ is the density with the congested flow $Q_e(\rho_{cg}(Q))$. The density of the forming vehicle queue is expected to be

$$\rho_{cg}(Q_{\text{bot}}) = \rho_{cg}(Q_{\text{max}} - \Delta Q),$$

(7.11)

which is expected to be close to $\rho_{max}$, corresponding to a high vehicle speed $V_e(\rho_{cg}(Q_{\text{bot}}))$. Kerner calls this state a *widening synchronized pattern* [?]. It can persist for a longer time period only, if the associated flow $Q_{\text{bot}}$ is stable. This requires that $\rho_{cg}(Q_{\text{bot}})$ lies in the metastable density range, i.e.

$$\rho_{c2} > \rho_{cg}(Q_{\text{bot}}),$$

(7.12)

which implies $\rho_{c2} > \rho_{max} = \rho_{fr}(Q_{\text{max}})$ and $Q_{c2} = Q_e(\rho_{c2}) < Q_{\text{bot}} = Q_{\text{max}} - \Delta Q$ or

$$\Delta Q < Q_{\text{max}} - Q_{c2}.$$  

(7.13)

If, however, $\rho_{c2} \leq \rho_{max} = \rho_{fr}(Q_{\text{max}})$, as has been assumed in many traffic simulations in the past, the vehicle queue upstream of the bottleneck is in an unstable flow regime, and the flow eventually breaks down. As a consequence, more serious congestion forms, and due to the capacity drop (see Sec. ...), the outflow from the bottleneck goes down from $Q_{\text{max}}$ to the value $\hat{Q} = Q_{\text{out}}$. This breakdown can be either triggered by

$$\Delta Q > Q_{\text{max}} - Q_{c2}$$

(7.14)

(i.e. a lack of capacity during rush hours) or by overcritical perturbations of the traffic flow (if $\rho_{cg}(Q_{\text{bot}})$ lies in the metastable regime). In the following, we will focus on the traffic states resulting after this breakdown of freeway capacity has taken place.
Fig. 7.3. WENJIAN, PLEASE INSERT A FIGURE HERE. Illustration of the situation before the breakdown of traffic flow (before the capacity drop). The diagram shows the upstream arrival flow $A$ over the bottleneck strength $\Delta Q$ per freeway lane. The diagonal lines represent traffic states of the same total traffic volume $Q_{\text{tot}} = A + \Delta Q$. Below the line $A = Q_{c1} - \Delta Q$, the overall traffic flow $Q_{\text{tot}}$ is always stable, i.e. one expects to find always free flow. Above this line, traffic flow is metastable upto the line $A = Q_{\text{max}} - \Delta Q$, which corresponds to the situation where the traffic volume $Q_{\text{tot}}$ reaches the freeway capacity $Q_{\text{max}}$. For $A > Q_{\text{max}} - \Delta Q$, queuing is unavoidable. Nevertheless, a capacity drop may not appear (for some time), if $\rho_{c2} > \rho_{\text{max}}$ and $\Delta Q < Q_{\text{max}} - Q_{c2}$, see the upper left triangle.

7.3.2 Conditions for Different Kinds of Congested Traffic after the Breakdown of Traffic Flow

For sake of simplicity, we will assume the case

$$Q_{c4} < Q_{c3} < Q_{c1} \leq Q_{\text{out}} \leq Q_{c2} < Q_{\text{max}},$$

which seems to be appropriate for real traffic (particularly in Germany). However, depending on the choice of model parameters, other cases are possible. The conclusions may be different, then, but the line of argumentation is the same. In the following we will furthermore assume $\rho_{c2} \geq \rho_{\text{max}}$, so that the maximum flow $Q_{\text{max}}$ would be metastable. Therefore, it could persist for some time, until the maximum flow state is destabilized by perturbations, and the traffic flow breaks down. The case $\rho_{c2} < \rho_{\text{max}}$ is discussed in Exercise 7....

After the breakdown of traffic flow, the traffic situation downstream is given by the outflow $\hat{Q} = Q_{\text{out}}$ from (seriously) congested traffic. As the entering ramp flow requires the capacity $\Delta Q$ per lane, the flow upstream of the bottleneck is limited to

$$Q_{\text{cong}} = Q_{\text{out}} - \Delta Q.$$

According to Eq. (??), the upstream front of this congested flow is expected to propagate with the velocity

$$C_2 = \frac{A - Q_{\text{cong}}}{\rho_{fr}(A) - \rho_{cg}(Q_{\text{cong}})},$$

while the downstream end would stay at the location of the bottleneck. The resulting congestion pattern depends on the stability properties of the vehicle density

$$\rho_{\text{cong}} = \rho_{cg}(Q_{\text{cong}}) = \rho_{cg}(Q_{\text{out}} - \Delta Q)$$

in the congested area. Therefore, let us now discuss the meaning of the critical densities or associated flows, respectively, within the phase diagram. We show the arrival flow $A$ upstream of the bottleneck over the bottleneck strength
\[ \Delta Q \text{ per lane. Lines of identical traffic volumes are falling diagonals (see Fig. ??).} \]

If \( \rho_{c2} < \rho_{\text{cong}} < \rho_{c3} \), we expect unstable, oscillatory traffic flow (OCT or SGW). For \( \rho_{c3} \leq \rho_{\text{cong}} < \rho_{c4} \), it is metastable, i.e. it depends on the perturbation amplitude: One may either have oscillatory patterns (for large enough perturbations) or homogeneous ones (for small perturbations). Moreover, for \( \rho_{\text{cong}} \geq \rho_{c4} \), we expect homogeneous, i.e. non-oscillatory traffic flows (if fluctuations in the data are smoothed away). Expressing these findings in terms of flows rather than densities, oscillatory congestion patterns (OCT or SGW) should be possible in the range

\[ 0 < \Delta Q < Q_{\text{out}} - Q_{c4}, \tag{7.19} \]

where we have considered \( Q_{c2} > Q_{\text{out}} \). Homogeneous traffic patterns should be possible in the ranges

\[ 0 < \Delta Q < Q_{\text{out}} - Q_{c1} \quad \text{and} \quad \Delta Q > Q_{\text{out}} - Q_{c3}. \tag{7.20} \]

Extended congested traffic requires the total flow \( Q_{\text{tot}} = A + \Delta Q \) to exceed the congested freeway capacity \( Q_{\text{out}} \), i.e.

\[ A > Q_{\text{out}} - \Delta Q = Q_{\text{cong}}. \tag{7.21} \]

However, congestion patterns can already be triggered for \( Q_{\text{tot}} > Q_{c1} \) (or even for \( Q_{\text{tot}} > Q_{c4} \)). That is, under the condition

\[ Q_{c1} - \Delta Q < \Delta Q < Q_{\text{out}} - \Delta Q, \tag{7.22} \]

one should for sure find localized traffic patterns. Their outflow would, of course, correspond to the inflow (otherwise they would grow or shrink in space). Therefore, the outflow of localized congestion patterns can be smaller than \( Q_{\text{out}} \), the outflow of serious congestion. At the same time we expect that traffic flows and vehicle speeds in localized congestion patterns may have values larger than zero.

We can distinguish at least two cases of localized congestion patterns: If

\[ Q_{c1} < A < Q_{c2}, \tag{7.23} \]

the flow upstream of the congested area is metastable, which allows jams (and large enough perturbations) to propagate upstream. In this case, we speak of moving localized clusters (MLC). Their propagation speed is expected to be around \( c \). Otherwise, for

\[ A \leq Q_{c1}, \tag{7.24} \]

localized congestion patterns cannot propagate upstream. In such cases, they should stay at the location of the bottleneck, and we call them pinned localized clusters (PLC).
We see that the instability diagram (??) implies a large degree of variability of congestion patterns despite of the simple simulation scenario of a homogeneous freeway with a ramp with varying inflows. If we would allow to change model parameters (corresponding to different driver behavior), the phase diagram may change considerably. Some examples are discussed in Exercises 7.

7.4 Pedestrian Flows at Bottlenecks

As a second example of inhomogeneous flows, let us now discuss the case of pedestrian flows at a bottleneck (e.g. exit or entrance gate). Doing so, we will follow the presentation of Ref. [?]. Due to the conservation of the number of pedestrians, we will describe the aggregate, two-dimensional pedestrian flow by the 2-dimensional continuity equation (??) with \( n = 2 \). Assuming a semicircular shape of a waiting crowd in front of the bottleneck suggest to express this equation in polar coordinates \( r \) and \( \theta \) via the relationship \( x = (x_1, x_2) = (r \cos \theta, r \sin \theta) \), where \( r \geq 0 \) denotes the distance from the bottleneck and \( \theta \) the angle with respect to the direction of motion. In polar coordinates, the continuity equation reads

\[
\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r \rho V) + \frac{1}{r} \frac{\partial}{\partial \theta} (\rho V \theta) = 0 \tag{7.25}
\]

(see Exercise 7...), where \( V \leq 0 \) denotes the average velocity towards the bottleneck and \( V \theta \) the velocity in perpendicular direction (which is related to a change in the angle \( \theta \)). Note that, for practical reasons, the direction of speed is opposite to the spatial coordinate \( r \), in contrast to our treatment of vehicle traffic. (This is, why we use \( r \) rather than \( x \), here.) In the following, we will assume that there is no significant dependence of the motion on the angle \( \theta \). Then, one simply obtains

\[
\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r \rho V) = 0, \tag{7.26}
\]

This equation can be rewritten as

\[
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho V)}{\partial r} = -\frac{\rho V}{r}, \tag{7.27} \{\text{contped}\}
\]

where the term on the right-hand side reflects a merging effect similar to the lane closure term (??) derived in the previous section. Specifically, this result corresponds to a spatial relation for the width of the kind \( I(r, t) = \pi r \) (the half circumference of a circle of radius \( r \)).

By use of logarithmic derivatives,\(^2\) Eq. (7.27) can also be presented in the form

\[^2\text{Here, we use the relationship } d \ln f(x)/dx = (df/dx)/f(x), \text{ which follows by application of the chain rule.}\]
\[
\frac{\partial \ln \rho(r,t)}{\partial t} = -V(r,t) \frac{\partial}{\partial r} \ln \left[ \rho(r,t)V(r,t) \right].
\]  
(7.28)

For the stationary case with \( \frac{\partial \ln \rho}{\partial t} = 0 \), it follows from \( \frac{\partial \ln(r \rho V)}{\partial r} = 0 \) that the overall flow \( \pi r \rho(r)V(r) \) through any cross section at distance \( r \) is constant:

\[
\pi r \rho(r)V(r) = \pi r Q(r) = -Q_0 = \text{const.} \quad (7.29)
\]

\( Q(r) = \rho(r)V(r) \) is the pedestrian flow through a cross section of unit length. \( Q_0 \geq 0 \) is the stationary overall pedestrian flow.

If we assume, for simplicity, a linear velocity-density relationship

\[
V(r) = V_0 \left( 1 - \frac{\rho(r)}{\rho_{\text{max}}} \right) \leq 0,
\]  
(7.30)

it is possible to derive analytical results. Here, \( V^0 \) means the maximum (“free”) pedestrian speed and \( \rho_{\text{max}} \) the maximum pedestrian density. Eqs. (7.29) and (7.30) give the quadratic equation

\[
\rho(r) V^0 \left[ 1 - \frac{\rho(r)}{\rho_{\text{max}}} \right] = \frac{Q_0}{\pi r}
\]  
(7.31)
in \( \rho \). With

\[
r_{\text{crit}}(Q_0) = \frac{Q_0}{(\pi Q_{\text{max}})}
\]  
(7.32)
it implies

\[
\rho_{\pm}(r, Q_0) = \frac{\rho_{\text{max}}}{2} \left( 1 \pm \sqrt{1 - \frac{r_{\text{crit}}(Q_0)}{r}} \right),
\]  
(7.33)

where

\[
Q_{\text{max}} = \frac{V^0 \rho_{\text{max}}}{4}
\]  
(7.34)
is the maximum flow. In free flow with \( d|\rho V_0(\rho)|/d\rho \geq 0 \), the density profile is determined by the upstream boundary condition, i.e. \( Q_0 \) is given by the overall inflow \( Q_{\text{in}} \). Under congested conditions \( d|\rho V_0(\rho)|/d\rho < 0 \), \( Q_0 \) is given by the overall outflow

\[
Q_{\text{out}} = \min(Q_{\text{in}}, 2r_0 Q_{\text{max}}),
\]  
(7.35)
i.e. the minimum of the overall inflow \( Q_{\text{in}} \) and the maximum possible overall outflow. The stationary case requires \( Q_{\text{in}} = Q_{\text{out}} \) and a non-negative discriminant in Eq. (7.33). This calls for \( r_0 \geq r_{\text{crit}}(Q_{\text{in}}) \), i.e. large outlets (see Fig. 12.1a). Then, \( \rho(r, t) \) converges to a stationary free flow with the density profile \( \rho_{\pm}(r, Q_{\text{in}}) \leq \rho_{\text{max}}/2 \). The density profile for other velocity-density relationships than (7.30) can be obtained numerically. Smooth perturbations like the humps in Fig. 12.1a propagate forward at the speed

\[
V_0(\rho) + \rho \frac{dV_0(\rho)}{d\rho} = -V^0 \left( 1 - 2 \frac{\rho}{\rho_{\text{max}}} \right)
\]  
(7.36)

(see Sec. ??), compactify close to the outlet and leave the system.
Fig. 7.4. Density profiles at different times, when (a) the inflow is low and the initial density profile has a hump, (b) the inflow exceeds the maximum outflow and the initial density profile is a step function (shock wave). The simulation results have been obtained by solving the continuity equation with the Godunov scheme, assuming $\rho(12r_{\text{crit}}, t) = 0.01$ and floating boundary conditions at $r = r_0 = 2r_{\text{crit}} := 2r_{\text{crit}}(Q_{\text{in}})$ in case (a), but $\rho(2.8r_{\text{crit}}, t) = 0.1$ and $Q_{\text{out}} = 2r_0 Q_{\text{max}}$ (corresponding to the maximum outflow) with $r_0 = 0.5r_{\text{crit}}$ in case (b). Note that the asymptotic density profile is $\rho_-(r, Q_{\text{in}})$ in free flow and $\rho_+(r, Q_{\text{out}})$ in jammed flow. (After Ref. \[?\].)
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If the overall inflow exceeds the overall outflow ($Q_{in} > Q_{out}$), pedestrians are jammed behind the outlet (Fig. 12.1b). The initial density profile $\rho(r,0)$ again approaches $\rho_{-}(r, Q_{in}) < \rho_{\text{max}}/2$ in the free-flow regime at large distances $r$, but converges to $\rho_{+}(r, Q_{out}) > \rho_{\text{max}}/2$ in the congested regime upstream of the outlet. The congestion front moves forward until the jam density $\rho_{+}(r, Q_{out})$ is reached. Then, the pedestrians queue up and the shock front at location $R(t)$ moves backward at the speed

$$\frac{dR}{dt} = -\frac{Q_{in} - Q_{out}}{\pi R(t)[\rho_{-}(R, Q_{in}) - \rho_{+}(R, Q_{out})]} \quad (7.37)$$

according to the shock wave equation (??) (see Fig. 12.1b). Hence, we find the free-flow density profile $\rho(r,t) \approx \rho_{-}(r, Q_{in})$ for $r > R(t)$, while for $r < R(t)$ we have the congested density profile $\rho(r,t) \approx \rho_{+}(r, Q_{out})$.

7.4.1 Intermittent Flows Due to Coordination Problems

The previous discussion applies to cases of continuous outflows, which are observed for large enough openings [??] or small enough pedestrian velocities $V^{0}$ [??]. However, if the desired velocity $V^{0}$ of pedestrians is high, their maximum density $\rho_{\text{max}}$ goes up and intermittent outflows are observed [??]. This intermittent behavior (see Fig. 7.6) shall be modeled now. For this, let us subdivide the pedestrian crowd into shells of thickness $\Delta r$ (for example, the pedestrian diameter $d$ or multiples of it). Within each shell of area $A(r) \approx \pi r \Delta r$, we assume a constant average density $\rho(r,t) = N(r,t)/A(r)$, where $N(r,t)$ denotes the number of pedestrians in the shell of radius $r$ at time $t$. Furthermore, we assume that pedestrians move from one shell to the next with velocity $V^{0}$, if they find a suitable gap, otherwise, they will stay. The maximum number of pedestrians available to move into the shell of radius $r$ is $\rho(r + \Delta r, t)A(r + \Delta r)$, while the maximum number of available gaps in shell $r$ is $\rho_{\text{max}}A(r)[1 - \rho(r,t)/\rho_{\text{max}}]$, because $\rho_{\text{max}}A(r)$ is the maximum number of pedestrians in the shell of radius $r$ and $Q(r,t) = 1 - \rho(r,t)/\rho_{\text{max}}$ represents the fraction of free space. Finally, we assume that $\xi_{+}^{r}Q(r,t)$ denotes the probability to find a suitable gap in front of a pedestrian allowing it to move, where $\xi_{+}^{r}$ are random numbers specified in each time step with $0 \leq \xi_{+}^{r} \leq 1$ and $\xi_{-}^{r} - \Delta r = \xi_{+}^{r}$ (in order to guarantee pedestrian conservation). Then, the number of inflowing pedestrians within the time interval $\Delta t = \Delta r/V^{0}$ is

$$\Delta N_{in}(r,t) = \xi_{+}^{r}Q(r,t) \min[A(r + \Delta r)\rho(r + \Delta r, t), A(r)\rho_{\text{max}}], \quad (7.38)$$

while the number of outflowing pedestrians is

$$\Delta N_{out}(r,t) = \xi_{-}^{r}Q(r - \Delta r, t) \min[A(r)\rho(r,t), A(r - \Delta r)\rho_{\text{max}}]. \quad (7.39)$$

Note that similar equations including minimum functions apply to merges of roads, see Eq. (??). From the conservation equation
\[ \Delta N(r, t + \Delta t) = \Delta N(r, t) + \Delta N_{in}(r, t) - \Delta N_{out}(r, t) \] 

and the density definition

\[ \rho(r, t) = \frac{\Delta N(r, t)}{A(r)} \]

we get

\[ \rho(r, t + \Delta t) = \rho(r, t) + \xi^+ r \left( 1 - \frac{\rho(r, t)}{\rho_{\text{max}}} \right) \min \left[ \left( 1 + \frac{\Delta r}{r} \right) \rho(r + \Delta r, t), \rho_{\text{max}} \right] \]

\[-\xi^- r \left( 1 - \frac{\rho(r - \Delta r, t)}{\rho_{\text{max}}} \right) \min \left[ \rho(r, t), \left( 1 - \frac{\Delta r}{r} \right) \rho_{\text{max}} \right]. \] 

Finally note that the half circle of radius \( r_0 \) around the exit is treated analogously to the shells, but we have to replace the area \( A(0) \) by \( A_0 = \pi r_0^2 / 2 \) and \( -N_{out}(0, t) \) by \(-2r_0 \rho_0(t) V^0 \Delta t \) (i.e. the exit width \( 2r_0 \) times the flow \( Q_0(t) = \rho_0(t) V_0 \), if pedestrians can leave with maximum velocity \( V_0 \) into the uncongested space behind the exit). The resulting equation for the density \( \rho_0(t) \) in the last (sub-)area before passing the bottleneck is

\[ \rho_0(t + \Delta t) = \rho_0(t) + \xi^+_0(t) \left( 1 - \frac{\rho_0(t)}{\rho_{\text{max}}} \right) \min \left[ \frac{2\Delta r}{r_0} \rho(r_0, t), \rho_{\text{max}} \right] - \frac{4\Delta r}{\pi r_0} \rho_0(t). \] 

The minimum function in Eq. (7.42) delineates the merging-related lack of space and outflow capacity. A similar situation and minimum function occurs in merging flows in street networks (see Sec. ??). With \( \Delta r = V^0 \Delta t, \rho = \rho(r, t), \xi_r = (\xi^+_r + \xi^-_r) / 2, \zeta_r = (\xi^+_r - \xi^-_r)(1 - \rho/\rho_{\text{max}}) / \Delta t \) and for \( \rho(r, t) \leq (1 - \Delta r / r) \rho_{\text{max}} \) we find the following equation in the limit \( \Delta t, \Delta r \to 0 \):

\[ \frac{\partial \rho}{\partial t} = V^0 \left( \xi^+_r - 2\xi^-_r \frac{\rho}{\rho_{\text{max}}} \right) \frac{\partial \rho}{\partial r} + \frac{\rho V^0 \xi^+_r}{r} \left( 1 - \frac{\rho}{\rho_{\text{max}}} \right) + \zeta_r \rho. \]

With the linear velocity-density relation (7.30), this exactly corresponds to the previous continuity equation (2.27), if \( \xi^\pm_r = 1 \), as for small enough densities (see below). Fluctuations \( \xi^\pm_r < 1 \), however, allow one to describe a dynamics in which less pedestrians than possible are successful in finding a gap in the next shell, because of coordination problems. The random variable \( \xi^\pm_r \) reflects that the microscopic spatial configuration of the pedestrians matters. When the second terms in the minimum functions of Eq. (7.42) apply, the dynamics is rather given by the equation

\[ \frac{\partial \rho}{\partial t} + V^0 \xi^-_r \frac{\partial \rho}{\partial r} = \frac{\rho V^0 \xi^-_r}{r} [\rho_{\text{max}} - \rho(r, t)] + \zeta_r \rho_{\text{max}}. \]

After averaging over the noise terms \( \xi^\pm_r \), representing the average of \( \xi^-_r \) by \( \bar{\xi} \), defining the the gap density \( \tilde{\rho}(r, t) = \rho_{\text{max}} - \rho(r, t) \), and introducing
\[ \hat{V} = V^0 \xi = \langle \xi_r^- \rangle, \]

(7.46)

this turns into a continuity equation for gap propagation:

\[ \frac{\partial \hat{\rho}}{\partial t} + \frac{\partial (\hat{\rho} \hat{V})}{\partial r} = -\frac{\hat{\rho} \hat{V}(t)}{r}. \]

(7.47)

Note that gaps propagate with velocity \( \hat{V} > 0 \), i.e. in opposite direction to the pedestrians.

We expect that a switching between gap propagation and pedestrian propagation by the minimum function can account for the intermittent outflows of dense granular flows. Triggered by the randomness of the variable \( \xi_r^\pm \), the switching mechanism can produce pedestrian avalanches of different sizes. The fluctuations \( \xi_r^\pm \) and their average value \( \bar{\xi} \) can be adjusted to experimental or suitable microsimulation results, e.g. to reflect the spatio-temporal fluctuations due to granular force networks. Here, we have instead simulated Eq. (7.42) with binomially distributed values of \( \xi_r^\pm \), i.e. \( \xi_r^\pm = k/N \) with

\[ P(k) = \binom{N}{k} p^k (1 - p)^{N-k}. \]

(7.48)

\( P(k) \) is the probability that \( k \in \{0,1,\ldots,N\} \) of \( N = N(r,t) \) pedestrians successfully manage to move forward, where \( p = \bar{\xi} \) is the probability of a pedestrian not to be obstructed. We have used the phenomenological specification

\[ p(\rho, r) = \left\{ 1 + \left[ \frac{r}{\Delta r} \left( \frac{\rho_{\text{max}}}{\rho} - 1 \right)^\beta + \epsilon \left( \frac{\gamma - \Delta r}{r} \right) \right]^{-1} \right\}^{-1} \]

(7.49)

if \( p > 0 \), otherwise \( p = 0 \). (\( \beta \), \( \gamma \), and \( \epsilon \) are non-negative fit parameters.) This ensures that \( p(\rho, r) \) becomes 1 for \( \rho \to 0 \) or \( r \to \infty \) and \( \max\{0, \epsilon(\gamma - \Delta r/r)\} \) for \( \rho \to \rho_{\text{max}} \). That is, we have complete clogging, if \( r_0/\Delta r < 1/\gamma \approx 5/2 \), which reflects arching if the outlet \( 2r_0 \) is too small (see Fig. 7.6). Otherwise, if the density is low or the bottleneck far away, we have \( p = 1 \) corresponding to pedestrians moving at the speed \( V = V_c(\rho) \). In queued areas with \( \rho \approx \rho_{\text{max}} \), gaps propagate upstream with velocity

\[ \hat{V} = V^0 \epsilon \left( \gamma - \frac{\Delta r}{r} \right). \]

(7.50)

### 7.4.2 Pinch Effect Close to Freeway Ramps

In summary, at bottlenecks we have alternating phases of forward pedestrian motion with speed \( V \) and of upstream dissolving pedestrian jams with average
Fig. 7.5. Top: Long-term photograph showing stop-and-go waves in a densely packed street. While stopped people appear relatively sharp, people moving from right to left have a fuzzy appearance. Note that gaps propagate from left to right. Middle: Empirically observed stop-and-go waves in front of the entrance to the Jammarat Bridge on January 12, 2006 (after [?]), where pilgrims moved from left to right. Dark areas correspond to phases of motion, light colors to stop phases. The “location” coordinate represents the distance to the beginning of the narrowing, i.e. to the cross section of reduced width. Bottom left: Illustration of the “shell model” (see Ref. [?]), in particular of situations where several pedestrians compete for the same gap, which causes coordination problems. Bottom right: Simulation results of the shell model. The observed stop-and-go waves result from the alternation of forward pedestrian motion and backward gap propagation.
speed $\hat{V}$, where $\hat{V}$ and $V$ may fluctuate in space and time. The formulas in the previous paragraphs are useful for the appropriate dimensioning of exits in order to avoid critical situations in cases of emergency evacuation of people. If the bottleneck is too small (and the desired speed $V^0$ of pedestrians too high), one may find intermittent flows close to and behind the bottleneck (Fig. 7.6 b, c). These are due to the fact that too many “particles” are competing for a confined space. Obviously, not all particles can successfully progress, when there are mutual obstructions. This “coordination problem” has been reflected by a fluctuation factor $\xi_{\pm}$, the mean value $p(\rho; I/(dI/dx))$ of which drops significantly below 1 if $\rho/\rho_{\text{max}} > 1 + \Delta x (dI/dx)/I(x)$.

The resulting dynamics is related to a stop-and-go phenomenon: In the high-density jam, the velocity is zero, as the pedestrians cannot move. However, jam resolution at the exit causes an upstream moving shock wave, in front of which the density is low. Therefore, pedestrians at the jam front can leave the jam. In fact, if the density in front of the jam is small enough, there is a forward motion of pedestrians filling the low-density area. Altogether, we will have alternating phases of jam resolution and gap filling processes close to the exit, which lead to alternating propagation directions of the jam front.

Analogous dynamical processes may explain the birth of small oscillatory perturbations in the so-called “pinch region” upstream of a freeway ramp [?], where lane-changes, especially by entering or exiting vehicles, cause coordination problems similar to the ones described above [?].

### 7.4.3 Relevance for Granular Flows in Hoppers

Intermittent flows are also well-known for granular materials [?, ?]. In fact, the above model can be easily adjusted to provide a simplified description of gravity-driven granular flows through a funnel (hopper). In such a case, granular particles replace the role of pedestrians, but cylindrical coordinates may often be applied. Because of the three-dimensional motion, this leads to a continuity equation of the form

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \rho v) = 0, \quad (7.51)$$

which can be rewritten as

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial r} = -\frac{2 \rho v}{r}, \quad (7.52)$$

or as

$$\frac{\partial \ln \rho(r, t)}{\partial t} = -v(r, t) \frac{\partial \ln [r^2 \rho(r, t) v(r, t)]}{\partial r} \quad (7.53)$$

when using logarithmic derivatives.
In order to avoid complicated boundary effects, it is reasonable to restrict the treatment to hoppers with walls that are steeper than the angle of repose. Therefore, the related opening angle is smaller than 180 degrees. In order to consider this, it is necessary to introduce a factor $f < 1$, which reflects what proportion of a half sphere is included in the opening of the funnel. We, therefore, have to use the relationship

$$ Q_{\text{in}} = 2f\pi r^2 \rho V(r) = 2f\pi r^2 Q(r) $$

and to replace $\pi r$ by $2f\pi r^2$ in the other formulas above. Then, the outflow is given by

$$ Q_{\text{out}} = \min(Q_{\text{in}}, \pi r_0^2 Q_{\text{max}}, 2f\pi r_0^2 Q_{\text{max}}). $$

That is, if $2f > 1$, the outflow is limited by the outlet of the funnel of size $2\pi r_0^2$, otherwise it is limited by the opening angle. The formula for the critical radius becomes

$$ r_{\text{crit}} = \sqrt{\frac{4c}{\xi_0 \rho_{\text{max}}}}. $$

Furthermore, the boundary condition in the three-dimensional shell model is

$$ \rho_0(t + \Delta t) = \rho_0(t) + \xi_0^+(t) \left(1 - \frac{\rho_0(t)}{\rho_{\text{max}}}ight) \min \left[\frac{d\Delta r}{r_0}\rho(r_0, t), \rho_{\text{max}}\right] - \frac{3\Delta r}{2f r_0} \rho_0(t), $$

and the equation for backward gap propagation reads

$$ \frac{\partial \hat{\rho}}{\partial t} + \frac{\partial (\hat{\rho}\hat{V})}{\partial r} = -\frac{2\hat{\rho}\hat{V}(t)}{r} $$

with

$$ \hat{V} = V^0 \epsilon + (1 - \epsilon) \left[1 + c \left(\frac{2 \Delta r/r}{1 - \rho/\rho_{\text{max}}}\right)^\beta\right]^{-1}. $$

The above model of intermittent flows due to coordination problems has an important feature that deserves mentioning. It does not only describe stop-and-go waves based on first-order traffic equations (see Fig. ??). It also implies a capacity drop (see Fig. ??). That is, the outflow grows proportionally with the inflow until the maximum flow .............. is reached. CAN IT BE ANALYTICALLY CALCULATED? However, instead of staying at this maximum value when the inflow is further increased, we find a drop of the capacity to another value that obviously characterizes the average outflow from stop-and-go traffic (see Fig. ??). This capacity drop is related to the coordination problems, which may be imagined to have a frictional effect.
Fig. 7.6. (a) In agreement with an experiment for granular outflows from a two-dimensional hopper [?], our simulation model generates exponentially distributed avalanche sizes when pedestrians with coordination problems are jamming at a bottleneck (i.e., theory and experiment show a straight line in a log-linear plot). (b) The standard deviation of the outflow, divided by the average outflow shows 3 regimes: no outflow for \( r_0/\Delta r < 1/\gamma = 2.5 \), smooth outflows for large outlets, and intermittent flow in between. (c) The relative proportion of time steps \( \Delta t \) with a stopped outflow confirms this picture. Our results are quite insensitive to the selected parameters. For illustration, we chose \( \beta = 3 \), \( \gamma = 2/5 \), \( \epsilon = 0.01 \), \( Q_{\text{in}} = 4/\Delta t \), \( V_0 = \Delta r/\Delta t \), \( \rho_{\text{max}} = 1/(\Delta r)^2 \) and, in (a), \( r_0 = 5 \Delta r \) (jammed conditions).

The existence of a capacity drop is well-known from vehicle and pedestrian flows. Its existence has serious consequences: If the maximum flow is exceeded, congestion sets in, as expected. But additionally, the capacity goes down. This makes congestion even worse, particularly during rush hours, when more capacity would be needed. As a consequence, congestion becomes even worse, which can lead to critical conditions in large pedestrian crowds [?]. In other words, it is essential to avoid the breakdown of capacity by limiting the inflow to the system. This calls for flow control, as it implies the validity of an old rule: “Less is more!”.

7.4.5 Empirical Evidence of the Transition to Stop-and-Go Waves

Recent empirical studies of pilgrim flows in the area of Makkah, Saudi Arabia, have shown that intermittent flows occur not only when bottlenecks are obvious. On January 12, 2006, pronounced stop-and-go waves have been even observed upstream of the 44 meter wide entrance to the Jamarat Bridge [?]. While the pilgrim flows were smooth and continuous (“laminar”) over many hours, at 11:53am stop-and-go waves suddenly appeared and propagated over distances of more than 30 meters (see Fig. 7.5). The sudden transition was related to a significant drop of the flow, i.e., with the onset of congestion [?]. Once the stop-and-go waves set in, they persisted over more than 20 minutes.

\[ \text{Fig2} \]
This phenomenon can be reproduced by a recent model based on two continuity equations, one for forward pedestrian motion and another one for backward gap propagation [?]. The model was derived from a “shell model” (see Fig. 7.5) and describes very well the observed alternation between backward gap propagation and forward pedestrian motion.

**7.4.6 Transition to “crowd turbulence”:**

**DISCUSSION IN THE CONTEXT OF BI-MODAL FUNDAMENTAL DIAGRAM, SEE COLOMBO ET AL.**

On the same day, around 12:19, the density reached even higher values and the video recordings showed a sudden transition from stop-and-go waves to *irregular* flows (see Fig. 7.7). These irregular flows were characterized by random, unintended displacements into all possible directions, which pushed people around. With a certain likelihood, this caused them to stumble. As the people behind were moved by the crowd as well and could not stop, fallen individuals were trampled, if they did not get back on their feet quickly enough. Tragically, the area of trampled people grew more and more in the course of time, as the fallen pilgrims became obstacles for others [?]. The result was one of the biggest crowd disasters in the history of pilgrimage.

**Fig. 7.7.** Pedestrian dynamics at different densities. Left: Representative trajectories (space-time plots) of pedestrians during the laminar, stop-and-go, and turbulent flow regime. Each trajectory extends over a range of 8 meters, while the time required for this stretch is normalized to 1. To indicate the different speeds, symbols are included in the curves every 5 seconds. While the laminar flow (top line) is fast and smooth, motion is temporarily interrupted in stop-and-go flow (medium line), and backward motion can occur in “turbulent” flows (bottom line). Right: Example of the temporal evolution of the velocity components $v_x(t)$ into the average direction of motion and $v_y(t)$ perpendicular to it in “turbulent flow”, which occurs when the crowd density is extreme. One can clearly see the irregular motion into all possible directions characterizing “crowd turbulence”. For details see Ref. [?].
How can we understand this transition to irregular crowd motion? A closer look at video recordings of the crowd reveals that, at this time, people were so densely packed that they were moved involuntarily by the crowd. This is reflected by random displacements into all possible directions. To distinguish these irregular flows from laminar and stop-and-go flows and due to their visual appearance, we will refer to them as “crowd turbulence”.

As in certain kinds of fluid flows, “turbulence” in crowds results from a sequence of instabilities in the flow pattern. Additionally, one finds a sharply peaked probability density function of velocity increments

\[ V_\tau^2 = V_x(r, t + \tau) - V_x(r, t), \text{ ADJUST VARIABLES } \quad (7.60) \]

which is typical for turbulence [?], if the time shift \( \tau \) is small enough [?]. One also observes a power-law scaling of the displacements indicating self-similar behaviour [?]. As large eddies are not detected, however, the similarity with fluid turbulence is limited, but there is still an analogy to turbulence at currency exchange markets [?]. Instead of vortex cascades like in turbulent fluids, one rather finds a hierarchical fragmentation dynamics: At extreme densities, individual motion is replaced by mass motion, but there is a stick-slip instability which leads to “rupture” when the stress in the crowd becomes too large. That is, the mass splits up into clusters of different sizes with strong velocity correlations inside and distance-dependent correlations between the clusters.

“Crowd turbulence” has further specific features [?]. Due to the physical contacts among people in extremely dense crowds, we expect commonalities with granular media. In fact, dense driven granular media may form density waves, while moving forward [?], and can display turbulent-like states [?]. Moreover, under quasi-static conditions [?], force chains [?] are building up, causing strong variations in the strengths and directions of local forces. As in earthquakes [?], this can lead to events of sudden, uncontrollable stress release with power-law distributed displacements. Such a power-law has also been discovered by video-based crowd analysis [?].

7.4.7 Some Warning Signs of Critical Crowd Conditions

Turbulent waves are experienced in dozens of crowd-intensive events each year all over the world [?]. Therefore, it is necessary to understand why, where and when potentially critical situations occur. Viewing real-time video recordings is not very suited to identify critical crowd conditions: While the average density rarely exceeds values of 6 persons per square meter, the local densities can reach almost twice as large values [?]. It has been found, however, that even evaluating the local densities is not enough to identify the critical times and locations precisely, which also applies to an analysis of the velocity field [?]. The decisive quantity is rather the “crowd pressure”, i.e. the density, multiplied with the variance of speeds. It allows one to identify critical locations and times (see Fig. 7.8).
Fig. 7.8. Left: Snapshot of the on-line visualization of “crowd pressure”. Red colors (see the lower ellipses) indicate areas of critical crowd conditions. In fact, the sad crowd disaster during the Muslim pilgrimage on January 12, 2006, started in this area. Right: The “crowd pressure” is a quantitative measure of the onset of “crowd turbulence”. The crowd disaster started when the “crowd pressure” reached particularly high values. For details see Ref. [?].

There are even advance warning signs of critical crowd conditions: The crowd accident on January 12, 2006 started about 10 minutes after “turbulent” crowd motion set in, i.e. after the “pressure” exceeded a value of $0.02/s^2$ (see Fig. 7.8). Moreover, it occurred more than 30 minutes after stop-and-go waves set in, which can be easily detected in accelerated surveillance videos. Such advance warning signs of critical crowd conditions can be evaluated on-line by an automated video analysis system. In many cases, this can help one to gain time for corrective measures like flow control, pressure-relief strategies, or the separation of crowds into blocks to stop the propagation of shockwaves [?]. Such anticipative crowd control could increase the level of safety during future mass events.

7.5 Exercises

1. What happens if $\rho_{c2} < \rho_{\text{max}}$, and what is the effectively resulting maximum flow that can persist over longer time periods?
2. Derive the phase diagram for models without critical thresholds $\rho_{c3}$ and $\rho_{c4}$. How does the phase diagram look for the Lighthill-Whitham-Richard model and the Burgers equation (see Exercise ...)? What is the situation in the Payne model and the optimal velocity model in cases of very small relaxation times $\tau$?
3. Derive the continuity equation ($??$) in polar coordinates from the 2-dimensional continuity equation with $x = (x_1, x_2) = (r \cos \theta, r \sin \theta)$ and $V = (V, V_{\theta})$. 
	\{$\text{turbul}$\}
7.6 Further Reading
8.1 Power Law Scaling of High-Flow States

It is natural that overtaking maneuvers of slow vehicles, e.g. overtaking trucks, constitute a moving bottleneck for faster vehicles for some time. These faster vehicles will be queued up behind the slow ones (the trucks) and form a platoon. When a vehicle platoon passes a cross section of the road, the traffic flow is particularly high due to the small time gaps between its vehicles. In the following, we are interested in the expected statistics of time periods of high flows.

The queue length will be given by the arrival rate of faster vehicles, i.e. basically by the upstream vehicle flow, together with the shock wave equation \( \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \). Intuitively, the number of vehicles in the platoon grows proportionally with the time required for the overtaking maneuver of the slow vehicles. Moreover, the period of high-flow states is expected to be proportional to the platoon length. In conclusion, the time period \( \Delta t' \) of high flow states is determined by the time period required for overtaking. If \( \Delta l_{\text{eff}}' \) denotes the effective distance over which an overtaking maneuver takes place and \( \Delta v \) the relative speed of the slow overtaking vehicle and the one it overtakes, we find the proportionality relation

\[
\Delta t' = \Delta l_{\text{eff}}' / \Delta v.
\]

As a consequence, the distribution of \( \Delta t' \) is given by the distribution of \( 1/\Delta v \).

Note that the speed distribution, which reflects the heterogeneity in the driving behavior, can be well approximated by a superposition of a normal distribution for trucks and a normal distribution for cars. As we need to determine the distribution of speed differences of slow vehicles, we can focus
on the truck distribution. From statistics, it is well-known that the difference of two identically, independently, and normally distributed variables is normally distributed as well. (EXERCISE) That is, the relative speeds $\Delta v$ of slow vehicles follow a normal distribution

$$N'(\Delta v) \, d\Delta v = \frac{1}{\sqrt{2\pi} \theta'} e^{-\left(\frac{\Delta v}{2\theta'}\right)^2}.$$  \hspace{1cm} (8.1)
From this fact we can derive the distribution of the variable \( y' = 1/\Delta v \) by application of the appropriate transformation. Considering

\[
\frac{dy'}{d\Delta v} = -\frac{1}{\Delta v^2} = -y'^2,
\]

we find

\[
N'(\Delta v) d\Delta v = -N'(\Delta v) \frac{1}{(y')^2} dy',
\]

where the minus sign is compensated for by integration from small to large values of \( y' = 1/\Delta v \) rather than vice versa. Therefore, we finally get the distribution

\[
P'(y') dy' = \frac{1}{\sqrt{2\pi \theta'}} \frac{1}{(y')^2} e^{-y'^2/(2\theta')^2} dy.
\]

In the limit of small speed differences \( \Delta v \), i.e. large values of \( y' \), we finally obtain the power law

\[
P'(y') \sim (y')^{-2},
\]

and, as \( y' = 1/\Delta v \sim \Delta t' \),

\[
P'(\Delta t') \sim (\Delta t')^{-2},
\]

Consequently, the time periods \( \Delta t' \) of high-flow states should be distributed according to a power law with exponent \(-2\). For larger values of \( y' \), the slope tends to be a bit steeper. This is well consistent with empirical observations [?].

### 8.2 Wide Scattering of Congested Flow-Density Data

According to Sec. ??, for a macroscopic traffic model assuming uniformly behaving driver-vehicle units with identical parameters, fully developed wide traffic jams should be characterized by a self-organized flow-density relation

\[
J(\rho) = \frac{1}{T} \left( 1 - \frac{\rho}{\rho_{\text{max}}} \right),
\]

where \( \rho \) denotes the vehicle density, while the average netto time gap \( T \) (the time clearance) and the maximum density \( \rho_{\text{max}} \) are assumed to be fixed parameters [?]. The dependence of the flow \( J \) on the density \( \rho \) in Eq. (8.7) is sometimes referred to as “jam line” \( J(\rho) \) and does not necessarily agree with the high-density part of the steady-state flow-density relation \( Q_f(\rho) \) for stationary and homogeneous traffic flow, the so-called fundamental diagram [?]. Moreover, the slope \( C = dJ(\rho)/d\rho = -1/(\rho_{\text{max}} T) \) is usually identified with the average propagation velocity of wide moving jams [?]. In contrast to wide moving jams, so-called “synchronized” traffic flow is a form of congested flow which is mostly found upstream of inhomogeneities (e.g. freeway bottlenecks)
and claimed to show a completely different behavior [?]. It is characterized by an erratic motion of time-dependent flow-density data in a two-dimensional area (and a synchronization of the time-dependent average vehicle velocities among neighboring lanes) [?]. More specifically, in synchronized flow, the slopes
\[ C(t_{k+1}) = \frac{Q(t_{k+1}) - Q(t_k)}{\rho(t_{k+1}) - \rho(t_k)} \]
(8.8)
of the lines connecting flow-density data measured at a given freeway cross section at successive times \( t_k \) and \( t_{k+1} \) erratically take on positive and negative values, characterizing a complex spatio-temporal dynamics [?].

This is, in fact, one of the most controversial subjects in traffic theory. It has not only been the reason for Kerner’s fundamental criticism of all traffic models assuming a fundamental (steady-state) relation \( Q_f(\rho) \) between the flow \( Q \) and the density \( \rho \) (i.e. of the vast majority of models suggested up to now) [?, ?]. It has also triggered a flood of publications with various proposals how to describe this wide scattering.

A particularly simple and successful explanation is based on the observed mixture of different vehicle types such as cars and trucks [?]. Let us check this by means of single-vehicle data, which were recorded by double induction-loop detectors. The measured data include, for each lane, the passage time \( t_i^0 \) of vehicle \( i \), its velocity \( v_i \), and its length \( l_i \). From this, we have determined the individual netto time gaps via the formula
\[ T_i = t_i^0 - t_{i-1}^0 - l_i / v_i. \]  
(8.9)
Furthermore, as discussed in Ref. [7], we have obtained macroscopic quantities by averaging over \( N = 50 \) successive vehicles, which gives a better statistics than averaging over a fixed time interval. Thus, we have obtained the empirical traffic flow at time \( t_k = \frac{1}{N} \sum_{i=(k-1)N+1}^{kN} t_i^0 \) by \( Q(t_k) = 1/(t_i^0 - t_{i-1}^0) = N/(t_k^0 - t_{(k-1)N}^0) \) and the average netto time gap by

\[
T(t_k) = \langle T_i \rangle = \frac{1}{N} \sum_{i=(k-1)N+1}^{kN} T_i = T^{(N)}_{(k-1)N}.
\] (8.10)

Likewise, we have defined the vehicle density \( \rho \) and the maximum density \( \rho_{\text{max}} \) by

\[
\frac{1}{\rho(t_k)} = \frac{1}{N} \sum_{i=(k-1)N+1}^{kN} v_i (t_i^0 - t_{i-1}^0).
\] (8.11)

\[
\frac{1}{\rho_{\text{max}}(t_k)} = \langle l_i \rangle = \frac{1}{N} \sum_{i=(k-1)N+1}^{kN} l_i.
\] (8.12)

For a given loop detector on the left lane, we have considered all time intervals with congested traffic (defined by \( \rho(t_k) \geq 45 \) vehicles per kilometer and lane) and compared the temporal changes \( \Delta Q^{(k+1)} = [Q(t_{k+1}) - Q(t_k)] \) of the empirically measured flow with the changes predicted by Eq. (8.7). We will compare two hypotheses by correlation analysis:

(i) \( \rho \) is treated as an independent variable defined by (8.11), while \( T \) and \( \rho_{\text{max}} \) are treated as parameters,

(ii) \( \rho \) and \( 1/T \) are independent variable defined directly from the single-vehicle data via (8.11) and (8.10), while \( \rho_{\text{max}} \) is a parameter.

The first order Taylor approximations for the temporal change \( \Delta J^{(k+1)}_{(\alpha)} = [J_{(\alpha)}(t_{k+1}) - J_{(\alpha)}(t_k)] \) of the corresponding theoretical vehicle flows are obtained by differential of relation (8.7) with respect to the independent variables, but not the parameters:

\[
\Delta J^{(k+1)}_{(i)} = -\frac{1}{\rho_{\text{max}} T} \rho(t_k) [\rho(t_{k+1}) - \rho(t_k)],
\] (8.13)

\[
\Delta J^{(k+1)}_{(ii)} = \Delta J^{(k+1)}_{(i)} + \left[1 - \frac{\rho(t_k)}{\rho_{\text{max}}} \right] \left[ \frac{1}{T(t_{k+1})} - \frac{1}{T(t_k)} \right],
\] (8.14)

Our statistical analysis gives the following correlations

\[
\text{Corr}(\Delta Q, \Delta J_{(\alpha)}) = \langle \sum_k \Delta Q^{(k)} \Delta J^{(k)}_{(\alpha)}/[\sum_k \Delta Q^{(k)} \sum_k \Delta J^{(k)}_{(\alpha)}] \rangle^{1/2}
\] (8.15)

for the hypotheses \( \alpha = i \) and \( ii \): \( \text{Corr}(\Delta Q, \Delta J_{(i)}) = 0.347 \), while \( \text{Corr}(\Delta Q, \Delta J_{(ii)}) = 0.918 \). As a consequence, hypothesis (i) assuming a linear variation of the flow
with the density is in fact a poor description of “synchronized” congested traffic data due to their two-dimensional scattering. However, hypothesis (ii) taking into account the dynamical variation of the average netto time gap \( T \) yields a very strong correlation, which indicates that we have identified the main reason for the wide scattering (see Fig. 8.3). Taking into account a variation of \( \rho_{\text{max}} \) with the truck fraction improves the correlation coefficient to 0.938.

It is obviously the variability of the time gaps, which explains the wide scattering of congested traffic flows, as suggested by Banks [?]. In order to further support this idea, let us calculate the variance of the sample-averaged time gaps as a function of the sampling size \( N \),

\[
\text{Var}(T) = \frac{1}{I-N+1} \sum_{j=0}^{I-N} \left( T_j^{(N)} - \bar{T} \right)^2 , \quad (8.16)
\]

where \( (j+1) \) runs over the first vehicle indices of all possible samples of \( N \) consecutive vehicles in the data set of size \( I \), \( T_j^{(N)} = \frac{1}{N} \sum_{i=j+1}^{j+N} T_i \) is defined as moving average of the time gaps of the \( N \) next vehicles, and \( \bar{T} = \sum_{i=1}^{I} T_i / I = T_0^{(I)} \) is the global average. For a given \( N \), the variance (8.16) decreases with increasing density, as less and less space is available for time gaps larger than the safe time gap, but the variance is also a function of the sampling size \( N \). For free traffic flow (with \( \rho < 15 \text{ veh/km/lane} \)), we observe the power law behavior

\[
\text{Var}(T) \propto \frac{1}{N} = N^{-1} , \quad (8.17)
\]

as expected for statistically independent time gaps in free traffic. At high vehicle densities (with \( \rho > 35 \text{ veh/km/lane} \)), however, we find the power law behavior

\[
\text{Var}(T) \propto N^{-0.67} \quad (8.18)
\]

[?]. That is, the relative variance decreases much slower with the sampling size than expected, implying that the time gaps do not average well and fluctuations of average time gaps remain significant for reasonable sampling sizes \( N \). This is related with surprisingly wide time gap distributions and results from correlations between queued vehicles, which are probably due to platoon formation (see Sec. ??).

### 8.3 Synchronization of Heterogeneous Speeds

One of the most controversially and intensively discussed subjects in traffic flow modeling in recent years was “synchronized traffic”. The term goes back to Boris Kerner [?], and its definition has been subject to changes over time. Simplifying matters, “synchronized traffic” means summarizes more or less all forms of congested traffic apart from wide moving jams. The interesting aspects of “synchronized flow” are
8.3 Synchronization of Heterogeneous Speeds

Fig. 8.3. The two-dimensional scattering of empirical flow-density data in synchronized traffic flow of high density $\rho \geq 45$ veh/km/lane (see (a)) is well reproduced by the jam relation (8.7), when not only the variation of the density $\rho$, but also the empirically measured variation of the average time gap $T$ and the maximum density $\rho_{\text{max}}$ is taken into account (see (b)). The similarity between the experimental data and relation (8.7) is partly because the density $\rho(t_k)$ plotted in (a) and (b) (the x-value) is determined with the same formula (8.11), but the agreement of the empirical flow $Q = 1/\left<T_i + l_i/v_i\right>$ and of the theoretical relation $J(\rho, T, \rho_{\text{max}}) = \left[1 - (\langle l_i \rangle / \langle v_i (T_i + l_i/v_i) \rangle) \right]/\langle T_i \rangle$ (the y-values) is not trivial, as even low-order approximations of these formulas differ. The pure density-dependence $J(\rho)$ (thick black line) is linear and cannot explain a two-dimensional scattering. However, variations of the average time gap $T$ change its slope $-1/(\rho_{\text{max}} T)$ (see arrows), and about 95% of the data are located between the thin lines $J(\rho, T \pm 2\Delta T, 1/l) = (1 - \rho l)/(T \pm 2\Delta T)$, where $l = 3.6$ m is the average vehicle length, $T = 2.25$ s the average time gap, and $\Delta T = 0.29$ s the standard deviation of $T$. The predicted form of this area is club-shaped, as demanded by Kerner [?].
1. spatio-temporal aspects of the transitions between free traffic, synchronized traffic, and wide moving jams (see Sec. ?? and Refs. [?, ?, ?]),
2. the wide scattering of flow-density data (see Sec. ?? and Refs. [?, ?, ?]),
3. the synchronization (homogenization) of speeds among neighboring lanes (see Refs. [?, ?, ?]).

In this section, we will focus on the last point. The speed synchronization, i.e. the more or less identical dynamics of the average velocities in the neighboring lanes observed in congested traffic flows, is particularly interesting for countries with heterogeneous vehicle speeds, i.e. where a fast lane and a slow lane (truck lane) can be clearly distinguished. For example, in Germany, slow vehicles may overtake, but are supposed to return to the slow (right) lane as soon as possible. The left lane is intended for overtaking. So, how is speed synchronization possible, if vehicles on one lane tend to move faster and on the other lane tend to move slower?

In order to answer this question, let us model the dynamics of neighboring lanes macroscopically, taking into account their interaction due to lane changes. For simplicity, we will restrict ourselves to the case of a two-lane freeway, but the generalization to freeways of any number of lanes is easily possible. If $1/\tau_1^+$ represents the average individual lane-changing rate from lane 1 to lane 2, the number of vehicles changing per unit time and unit space performing such a lane change is $\rho_1/\tau_1^+$: It must obviously be proportional to the number of vehicles, which may change (each with a rate of $1/\tau_1^+$), i.e. proportional to the density $\rho_1$ in lane 1. Similarly, if $\tau_2^−$ is the lane-changing rate from lane 2 to 1, $\rho_2/\tau_2^−$ reflects the number of vehicles per unit time and unit space changing from lane 2 to 1.

The changes to and away from a lane determine the source and sink terms in the continuity equations describing the evolution of the vehicle density in each lane, see Eq. (??). In fact, we have to add the number of lane changes per unit time and unit length changing to the respective lane, and subtract the corresponding number of lane changes away from this lane. As a consequence, if we represent the lane number by the index of the vehicle density $\rho$ and the average velocity $V$, the macroscopic equations for the vehicle densities read:

$$\frac{\partial \rho_1}{\partial t} + \frac{\partial (\rho_1 V_1)}{\partial x} = \frac{\rho_2}{\tau_2} - \frac{\rho_1}{\tau_1^+},$$
$$\frac{\partial \rho_2}{\partial t} + \frac{\partial (\rho_2 V_2)}{\partial x} = \frac{\rho_1}{\tau_1^+} - \frac{\rho_2}{\tau_2}. \quad (8.19)$$

Adding both equations and defining the average density as $\rho = (\rho_1 + \rho_2)/2$ and the average flow as $Q = (Q_1 + Q_2)/2$, we obtain again the well-known continuity equation (?) for the overall cross section of the road, which contains no sink or source terms (if there are no changes in the number of lanes, or entering or exiting ramp flows). Therefore, the equations fulfil the requirement of conserving the number of vehicles (see Sec. ...).
In the following, we are particularly interested in the stationary and homogeneous case, where the temporal and spatial derivative vanish. For this case, we find
\[
\frac{\rho_1}{\tau_1} + \frac{1}{\tau_2} = \rho_2 \tau - \frac{2}{\tau_2} = R. \tag{8.20}
\]
This means the average flows of vehicles entering and exiting a lane due to lane changes are balanced. Otherwise, the density in one lane would grow due to lane changes, and in the other lane it would go down, until an equilibrium would eventually be reached. It is expected, that relation (8.20) is also approximately valid in cases of non-stationary and non-homogeneous flows, if the lane-changing rates are large enough (in comparison with the rate of change due to the longitudinal traffic dynamics in a lane).

As indicated before, lane 2 tends to be faster than lane 1. In other words, the respective stationary speed-density relations \(V^e_i(\rho_i)\) in absence of lane changes are different, as lane 2 tends to be used by faster (and shorter) vehicles, while lane 1 tends to be used by slower vehicles. We will, therefore, assume
\[
V^e_2(\rho) > V^e_1(\rho). \tag{8.21}
\]
So, how can the consideration of lane changes lead to speed synchronization, then? To explain this effect, we have to consider that overtaking vehicles are always a bit faster than overtaken vehicles, i.e. there is a transfer of momentum (or average speed) from one lane to the other, whenever a lane change takes place. The average effect of the momentum transfer is naturally proportional to the speed difference between lanes, i.e. \(\pm R(V_2 - V_1)\). This is to be added to the lane-specific speed relaxation term \(\rho_i/\tau(V^e_i - V_i)\) appearing in the equation for \(\rho_i \partial V_i/\partial t\), see Eq. (??). In the stationary and homogeneous case this implies
\[
\frac{\rho_1}{\tau_1} \left[ V^e_1(\rho_1) - V_1 \right] + R(V_2 - V_1) = 0,
\]
\[
\frac{\rho_2}{\tau_2} \left[ V^e_2(\rho_2) - V_2 \right] + R(V_1 - V_2) = 0. \tag{8.22}
\]
For the derivation of full dynamical equations for the average velocities in neighboring lanes, considering lane changing effects, see Refs. [?, ?].

Note that the relationships (8.22) are coupled equations. By rearranging terms we get
\[
\left( R + \frac{\rho_1}{\tau} \right) V_1 = \frac{\rho_1 V^e_1(\rho_1)}{\tau} + RV_2, \tag{8.23}
\]
\[
\left( R + \frac{\rho_2}{\tau} \right) V_2 = \frac{\rho_2 V^e_2(\rho_2)}{\tau} + RV_1. \tag{8.24}
\]
Inserting one equation into the other gives
This can be interpreted as weighted mean values of $V_e^1$ and $V_e^2$, considering the fact that

$$R - \frac{R^2}{R + \frac{\rho_1}{\tau}} = R \left( 1 - \frac{1}{1 + \frac{\rho_1}{\tau R}} \right) = \frac{\rho_1}{1 + \frac{\rho_1}{\tau R}} > 0. \quad (8.26)$$

From the previous relationships, it is straightforward to derive

$$V_2 - V_1 = \frac{\rho_2 \rho_1}{\rho_1 + \rho_2 + \rho_2 \frac{\rho_1}{\tau R}} \frac{V_e^2(\rho_2) - V_e^1(\rho_1)}{1 + \frac{\rho_1}{\tau R}}. \quad (8.27)$$

This relationship should hold exactly in the stationary and homogeneous case, otherwise it should still be approximately valid. It shows rather clearly that speed synchronization of both lanes, i.e. $V_2 = V_1$ is only possible for

1. $\tau R \to 0$, which is unrealistic,
2. $(\rho_1 + \rho_2) \to \infty$, which is also not the case we are interested in, or
3. $[V_e^2(\rho_2) - V_e^1(\rho_1)] = 0$.

Let us focus on the third case and consider $\rho = (\rho_1 + \rho_2)/2$ and $\rho_2 = (2\rho - \rho_1)$, which implies

$$V_2 - V_1 = \frac{V_e^2(2\rho - \rho_1) - V_e^1(\rho_1)}{1 + \frac{2\rho}{\tau R}}. \quad (8.28)$$

For this to become approximately zero, we need to have

$$V_e^2(2\rho - \rho_1) \approx V_e^1(\rho_1). \quad (8.29)$$

Considering our assumption $V_e^2(\rho) > V_e^1(\rho)$, this can only be the case if the density difference $\rho_2 - \rho_1 = 2(\rho - \rho_1)$ between both lanes becomes so large that the natural speed difference between both lanes is compensated. This can obviously only happen for densities $\rho_2$ for which $V_e^2(\rho_2) \leq V_e^1(0) = V_0$, i.e. not for small densities $\rho = (\rho_1 + \rho_2)/2$. For this reason, speed synchronization on freeways with fast and slow lanes is not found for free traffic flow. But even for medium and high densities, it requires a mechanism that causes a higher density in the fast lane, when traffic becomes congested. From Eq. (8.20) we can conclude

$$\frac{\rho_2}{\rho_1} = \frac{\tau_2}{\tau_1} \quad \text{and} \quad \rho_1 = \frac{2\rho}{\tau_1 + 1}, \quad (8.30)$$

which finally implies
\[ \rho_2 - \rho_1 = 2\rho \frac{\tau_2^-}{\tau_1^+} - 1 \frac{\tau_2^-}{\tau_1^+} + 1. \tag{8.31} \]

Speed synchronization is understandable if the average individual lane changing rate \(1/\tau_1^+\) exceeds the rate \(1/\tau_2^-\) by a sufficient amount. Therefore, we have to focus on the specification of the lane changing rates. The lane changing rate \(1/\tau_2^-\) from the fast to the slow lane is mainly determined by the traffic regulation requiring to get back to the right lane after overtaking (as long as traffic flows freely). It is, therefore, expected to **decrease** as the speed difference \(V_2 - V_1\) goes down, i.e. as the vehicle density goes up. For \(V_2 = V_1\), this term should even become zero or at least very small. In contrast, the lane changing rate from the slow to the fast lane is expected to increase with the interaction rate and, therefore, with the vehicle density (at least as long as there are opportunities and incentives to overtake, in particular as long as \(V_2 > V_1\)). Since the rate \(1/\tau_2^-\) goes down and \(1/\tau_1^-\) goes up with growing density (as long as densities are not too high to find suitable gaps for lane changes), \(\frac{\tau_2^-}{\tau_1^+}\) eventually exceeds 1 and becomes large enough to equilibrate the speeds. Only when this has happened, there is no incentive anymore for additional vehicles to change to lane 2. As a consequence, the increase of the vehicle density on lane 2 stops (exactly), when the speed \(V_2\) in lane 2 has fallen down to the value \(V_1\) in lane 1. That is the explanation of speed synchronization. For numerical investigations of the synchronization phenomenon, see e.g. Refs. [? , ? ]. According to these, as soon as speed synchronization is established, the overall number of lane changes drops significantly due to missing incentives. Note, however, the effect of speed variances, which never lets the lane changing rates drop to zero [?].
One of the most interesting findings in systems governed by non-linear partial differential equations or non-linear sets of ordinary differential equations is self-organization. This should particularly apply to driven many-particle systems, traffic flows, pedestrian crowds, etc. In fact, we have already discussed stop-and-go traffic, the jam line, and characteristic constants as results of self-organization (see Secs. ...).

Self-organization means the spontaneous formation of spatial and/or temporal patterns (other than stationary and homogeneous solutions) as a result of non-linear interactions in the system under consideration. While these patterns may change as a function of control parameters, initial, or boundary conditions, it is important to state that they are not externally determined or enforced by those. That is, the observed patterns would not follow from the initial or boundary conditions themselves, which are often unspecific. The spatial and/or temporal dependence of the patterns is often much more complicated than the initial or boundary conditions, and they cannot be understood without the consideration of the non-linear interactions in the system.

For illustration, just remember the formation of oscillatory congested traffic (see Sec. ...). According to the phase diagram displayed in Fig. ..., we expect a transition from free traffic to an extended, oscillatory kind of congested traffic when the total traffic volume $Q_{tot}$, which plays the role of the control parameter here, exceeds a certain critical threshold. This should even hold if the upstream and on-ramp flows would not show oscillations at all. Therefore, the oscillations can only be a result of self-organization, not of the boundary conditions (the inflows) or any external influence factors. In fact, they are a consequence of instabilities resulting from the interactions of its elements.\footnote{Note that the instability itself does not require non-linear interactions, but the resulting patterns do. Unstable behavior based on linear interactions would imply ever-growing wave amplitudes, so that we would not find a stabilizing pattern after...}
The creation of novelty or, more precisely, the appearance of new and often unexpected kinds of patterns is a typical feature of complex systems and called *emergence*. As indicated above, emergent phenomena cannot be understood as a superposition of the behavior of the system ingredients. In systems characterized by non-linear interactions, one can often find a large variety of patterns, when one or a few control parameters are varied. The phase diagram discussed in Sec. ?? is a good illustration. Therefore, the same kinds of interactions can produce very different outcomes, as control parameters are varied, even if the interactions are very simple (as for the optimal velocity or the social force model). In such cases, one typically talks of complex systems.

The phase diagram of Sec. ?? also demonstrates another typical feature of complex systems. When control parameters such as the total traffic volume are varied, within a certain range, the resulting system behavior changes only quantitatively, while the qualitative appearance of the pattern (i.e. the kind of congestion) stays the same. However, when a control parameter crosses a so-called critical threshold, one finds a sudden change of the pattern and, therefore, the system behavior. Such abrupt (phase) transitions have been studied by the disciplines of catastrophe theory [?] and statistical physics [?]. But sudden changes are not necessarily catastrophic. Often, they reflect the capability of complex systems to adapt to changing conditions (represented by changing control parameters). And only super-critical changes cause a transition in system behavior, while the system tends to be robust with respect to small enough variations. Such features are well-known from biological systems and very essential to understand life [?]. Systems' theory is trying to grasp and formalize these properties [?, ?]. More recently, they are used to design new, decentralized control concepts. In contrast to top-down control, the idea is to make use of spontaneous self-organization in the system, i.e. to apply a bottom-up approach. It is important that these self-organization processes can be “guided” by small modifications of the interactions [?]. Examples are traffic assistance systems based on cooperative driving [?] and self-organizing traffic lights [?].

Finally, let us discuss what happens at the critical thresholds. Here, fluctuations can have a dominating influence on the system behavior (while they are damped out in the parameter ranges characterized by robust system behavior). As a consequence of the dominating fluctuations, one typically finds fat tail distributions such as power laws rather than normal distributions. Therefore, extreme events (large fluctuations) occur much more frequent than a normal distribution would predict. A typical example are bubbles and crashes at stock markets [?]. The formal description of such phenomena is addressed by the theory of critical phenomena [?]. It often allows one to derive the characteristics of fluctuations (e.g. the power law exponents) and scaling laws for the system behavior close to a critical threshold. Note that some systems have some time. The stabilization is a result of non-linearities which, for example, limit the growth of the amplitudes.
mechanisms that automatically drive it towards a critical threshold. In such cases, one speaks about self-organized criticality (SOC) [?].

9.1 Self-Organization Phenomena in Pedestrian Crowds

When simulating the social force model of pedestrian dynamics on a computer, one can observe various self-organization phenomena, depending on the respective situation. In spite of the simplicity of the model, it reproduces the empirical observations quite realistically. The social force model particularly allows one to explain various self-organized spatio-temporal patterns that are not externally planned, prescribed, or organized, e.g. by traffic signs, laws, or behavioral conventions. Instead, the spatio-temporal patterns discussed below emerge due to the non-linear interactions of pedestrians even without assuming strategical considerations or communication of pedestrians. Many of these collective patterns of motion are symmetry-breaking phenomena that occur even if we assume completely symmetric behavior with respect to the right-hand and the left-hand side [?, ?, ?, ?].

9.1.1 Lane Formation

Our microsimulations reproduce the empirically observed formation of lanes consisting of pedestrians with the same desired walking direction (see Fig. 9.1). For open boundary conditions, these lanes are dynamically varying. Their number depends on the width of the street, on pedestrian density, and on the variability of speeds.

The conventional interpretation of lane formation assumes that pedestrians tend to walk on the side which is prescribed in vehicular traffic. However, the above model can explain lane formation even without assuming a preference for any side. The most relevant point is the higher relative velocity of pedestrians walking in opposite directions. Pedestrians moving against the stream or in areas of mixed directions of motion will have frequent and strong interactions. In each interaction, the encountering pedestrians move a little aside in order to pass each other. This sideways movement tends to separate oppositely moving pedestrians, which leads to segregation. The resulting collective pattern of motion minimizes the frequency and strength of avoidance maneuvers, if fluctuations are weak. Assuming identical desired velocities $v_i^0 = v_0$, the most stable configuration corresponds to a state with a minimization of the average interaction strength

$$-\frac{1}{N} \sum_{i \neq j} \tau_{ij} \cdot e_i^0 \approx \frac{1}{N} \sum_i (v^0 - v_i \cdot e_i^0) = v^0 (1 - E). \quad (9.1)$$

EXERCISE: DERIVE THIS DIRECTLY FROM THE SOCIAL FORCE MODEL. WHAT APPROXIMATIONS ARE MADE? It is related with a maximization of the efficiency
Fig. 9.1. Formation of lanes in initially disordered pedestrian crowds with opposite walking directions. Arrows represent the velocities of the pedestrians. Note that lane formation does not require the periodic boundary conditions applied above, see the Java applet http://www.helbing.org/Pedestrians/Corridor.html.

\[
E = \frac{1}{N} \sum_i \frac{v_i \cdot e_0^i}{v_0} \tag{9.2} \]

of motion, where the efficiency \( E \) with \( 0 \leq E \leq 1 \) describes the average fraction of the desired speed \( v^0 \) with which pedestrians actually approach their destinations (\( N = \sum_1 \) is the respective number of pedestrians \( i \)). As a consequence, lane formation “globally” maximizes the average velocity into the respectively desired direction of motion, although the model does not even assume that pedestrians would try to optimize their behavior locally. This is a consequence of the symmetrical interactions among pedestrians with opposite walking directions. One can even show that a large class of driven many-particle systems, if they self-organize at all, tend to globally optimize their state (see Sec. ??).

9.1.2 Oscillations at Bottlenecks

In simulations of bottlenecks like doors, we often observe oscillatory changes of the passing direction (see Fig. 9.2). Once a pedestrian is able to pass the narrowing, pedestrians with the same walking direction can easily follow. Hence,
the number and “pressure” of waiting and pushing pedestrians becomes less than on the other side of the narrowing where, consequently, the chance to occupy the passage grows. This leads to a deadlock situation which is followed by a change in the passing direction.

Fig. 9.2. Oscillations of the passing direction at a bottleneck. Dynamic simulations are available at [http://www.helbing.org/Pedestrians/Door.html](http://www.helbing.org/Pedestrians/Door.html).

The resulting oscillatory pedestrian flows at narrow bottlenecks almost look as if they were controlled by traffic lights. This has inspired the invention of a new, decentralized traffic light control [?], since intersections are like bottlenecks for two or often more competing traffic flows.

### 9.1.3 Pedestrian Flows at Intersections

The dynamics of crossing pedestrian flows is even more complex than that of traffic flows. At intersections one is confronted with various alternating collective patterns of motion which are very short-lived and unstable. For example, phases during which the intersection is crossed in “vertical” or “horizontal” direction alternate with phases of temporary roundabout traffic (see Fig. ??). This self-organized round-about traffic is similar to the emergent rotation found for self-driven particles [?]. It is connected with small detours but decreases the frequency of necessary deceleration, stopping, and avoidance.
maneuvers considerably, so that pedestrian motion becomes more efficient on average.

![Diagram of intersecting pedestrian streams with stripes]

Fig. 9.3. Self-organized, short-lived roundabout traffic in intersecting pedestrian streams.

### 9.1.4 Stripe Formation in Two Intersecting Flows

In intersection areas, the flow of people often appears to be irregular or “chaotic”. In fact, it can be shown that there are several possible collective patterns of motion, among them rotary and oscillating flows. However, these patterns continuously compete with each other, and a dominating patterns is destroyed by another one after a short time. Obviously, there has not evolved any social convention that would establish and stabilize an ordered and efficient flow.

Self-organized patterns of motion, however, are observed in situations where pedestrian flows intersect each other only in two directions. In such situations, the phenomenon of stripe formation is observed [?]. Stripe formation allows two flows to penetrate each other without requiring the pedestrians to stop. For an illustration see Fig. 9.4. Like lanes, stripes are a segregation phenomenon, but not a stationary one. Instead, the stripes are density waves moving into the direction of the sum of the directional vectors of both intersecting flows. Naturally, the stripes extend sidewards into the direction which
is perpendicular to their direction of motion. Therefore, the pedestrians move forward with the stripes and sidewards within the stripes. Lane formation corresponds to the particular case of stripe formation where both directions are exactly opposite. In this case, no intersection takes place, and the stripes do not move systematically. As in lane formation, stripe formation allows to minimize obstructing interactions and to maximize the average pedestrian speeds, i.e. simple, repulsive pedestrian interactions again lead to an “intelligent” collective behavior. This can actually be used to pedestrian intersections more efficiently: Rather than allowing four flow directions to cross each other in one area, it is better to separate opposite flows and to organize four intersections of two directions each (see Fig. ...). If properly done, this creates more orderly flows and less mutual obstructions [?].

Fig. 9.4. Simulation of two crossing pedestrian streams, showing the phenomenon of stripe formation. This self-organized pattern allows pedestrian to pass the other stream without having to stop, namely by moving sidewards in a forwardly moving stripe.
9.2 A Simple Model for the Success-Driven Self-Organization in Space

9.2.1 Derivation of the Model for Opposite Pedestrian Flows

In Sec. ??, we could show that the segregation of opposite directions of motion into lanes minimizes the interaction intensity and related friction or dissipation effects. Let us now set up a model for this segregation effect in a corridor. \( \rho_a(x, t) \) shall represent the spatial density of pedestrians with a desired walking direction \( a \), and \( \mathbf{V}_a(x, t) = (V_{a,1}, V_{a,2}) \) their average velocity at place \( x = (x_1, x_2) \). \( V_{a,k} \) represent the velocity components in the direction \( x_1 \) along the corridor and \( x_2 \) perpendicular to it. Assuming for simplicity a circular corridor with periodic boundary conditions and no change between clockwise motion (\( a = 1 \)) and counter-clockwise motion (\( a = 2 \)), the number \( N_a = \int d^2 x \rho_a(x, t) \) (9.3) of pedestrians in each of the two possible streams \( a \in \{1, 2\} \) of motion is constant. Moreover, according to Exercise ..., we can apply the continuity equation

\[
\frac{\partial \rho_a(x, t)}{\partial t} + \sum_{k=1}^{2} \frac{\partial}{\partial x_k} [\rho_a(x, t)V_{a,k}(x, t)] = 0
\]

(9.4)

for \( a \in \{1, 2\} \). Let us furthermore assume a homogeneous distribution of pedestrians along the direction \( x_1 \) of the corridor. Then, with the abbreviations \( x = x_2 \) and \( V_a = V_{a,2} \), we stay with the one-dimensional continuity equation

\[
\frac{\partial \rho'_a(x, t)}{\partial t} + \frac{\partial}{\partial x} [\rho'_a(x, t)V_a(x, t)] = 0,
\]

(9.5)

where the \( \rho'_a(x, t) \) denotes the pedestrian density per width. If \( I \) denotes the overall width, we have the relation

\[
N_a = \int_0^I dx \rho'_a(x, t).
\]

(9.6)

In the following, the prime (') will be dropped for simplicity of notation.

Let us now derive a relationship for the average velocity \( V_a(x, t) \) of pedestrian flow \( a \) perpendicular to the walking direction: First, it is expected to be proportional to the interaction frequency \( \nu_a \) with other pedestrians that a pedestrian belonging to flow \( a \) encounters. Second, it should be proportional to the average amount \( \Delta x \) that a pedestrian moves aside when evading another pedestrian. Third, it will be proportional to the difference of the probabilities \( p_+ \) and \( p_- \) to move in positive or negative \( x \)-direction, respectively. In summary, we have the relation
9.2 A Simple Model for the Success-Driven Self-Organization in Space

\[ V_a(x,t) = c \nu_a \Delta x (p_+ - p_-), \quad (9.7) \]

where we have neglected fluctuations. The interaction rate of pedestrians belonging to (sub-)population \( a \) with others is

\[ \nu_a = \sum_{b=1}^{2} C_{ab} \rho_b, \quad (9.8) \]

where \( C_{ba} > C_{aa} \) because of the higher average relative velocity between oppositely moving pedestrians. For details, see Refs. [?, ?].

We will assume that the probability of moving by \( \Delta x \) in positive (or negative) \( x \)-direction, when evading a pedestrian, is inversely proportional to the interaction rate at position \( x_+ = x + \Delta x \) (or \( x_- = x - \Delta x \), respectively):

\[ p_+ = \frac{1/\nu_a(x_+)}{1/\nu_a(x_+)} = \frac{1/\nu_a(x_-)}{1/\nu_a(x_+)} = \frac{1/\nu_a(x_-)}{1/\nu_a(x_-)}. \quad (9.9) \]

A first order Taylor expansion of the nominator and the denominator gives the following approximate relation for the difference of these probabilities:

\[ (p_+ - p_-) \approx -\frac{\Delta x}{\nu_a(x)} \frac{\partial \nu_a(x)}{\partial x}. \quad (9.10) \]

(which, strictly speaking, is restricted to cases of small gradients \( \partial \nu_a/\partial x \) as in the linear regime around the homogeneous solution). Hence, with (9.7) and (9.8), we finally obtain the following proportionality for the average velocity of motion perpendicular to the desired walking direction:

\[ V_a(x,t) \sim - (\Delta x)^2 \sum_{b=1}^{2} C_{ab} \frac{\partial \rho_b}{\partial x}. \quad (9.11) \]

Denoting the proportionality factor by \( c \), defining \( P_{ab} = -c(\Delta x)^2 C_{ab} \) and generalizing to an arbitrary number \( A \) of (sub-)populations \( b \) gives

\[ V_a(x,t) = \sum_{b=1}^{A} P_{ab} \frac{\partial \rho_b}{\partial x}. \quad (9.12) \]

We may rewrite this in terms of a gradient

\[ V_a(x,t) = \frac{\partial S_a(x,t)}{\partial x}. \quad (9.13) \]

of the linear, density-dependent “success function”

\[ S_a(x,t) = \sum_b P_{ab} \rho_b(x,t). \quad (9.14) \]
For higher-dimensional spaces, relation (9.13) becomes a potential condition. This is one of the conditions which must be fulfilled for optimal self-organization (see Sec. ...).

Finally, let us take into account fluctuations $\xi_a(t)$ of the success function $S_a(x,t)$. The more general specification

$$S_a(x,t) = \sum_b P_{ab} \rho_b(x,t) + \xi_a(t)$$  \hspace{1cm} (9.15) \hspace{1cm} \{\text{success2}\}

finally leads to an additional, fluctuation-related diffusion term (see Sec. ...). This generalizes the continuity equation (9.5) to give

$$\frac{\partial \rho_a(x,t)}{\partial t} + \frac{\partial}{\partial x} \left[ \rho_a(x,t) V_a(x,t) \right] = \frac{\partial}{\partial x} \left( D_a \frac{\partial \rho_a(x,t)}{\partial x} \right).$$  \hspace{1cm} (9.16)

(EXERCISE) The diffusion coefficient $D_a$ is proportional to the standard deviation $\rho_a$ of $\xi_a(t)$. For details see Ref. [?]. It is interesting to note that the effect of diffusion corresponds to an effective average velocity of

$$V_a(x,t) = \frac{\partial S_a}{\partial x} - \frac{D_a}{\rho_a} \frac{\partial \rho_a}{\partial x} = \sum_b P_{ab} \frac{\partial \rho_b}{\partial x} - D_a \frac{\partial \ln \rho_a}{\partial x}.$$  \hspace{1cm} (9.17)

$D_a$ is not necessarily a constant. It may be specified in a density-dependent way.

9.2.2 Self-Optimization

In the following, we will prove that, for vanishing diffusion ($D_a = 0$) and under some additional conditions, the function

$$S(t) = \sum_a \int dx \ rho_a(x,t) S_a(x,t) = \sum_a \int dx \sum_b P_{ab} \rho_b(x,t)$$  \hspace{1cm} (9.18) \hspace{1cm} \{define\}

is a so-called Lyapunov function, which increases monotonously in the course of time.$^2$

By deriving (9.18) with respect to $t$ and using (9.40), one obtains

$$\frac{dS(t)}{dt} = \sum_a \int dx \left( \frac{\partial \rho_a(x,t)}{\partial t} \sum_b S_{ab} \rho_b(x,t) + \rho_a(x,t) \sum_b P_{ab} \rho_b(x,t) \right).$$  \hspace{1cm} (9.19)

Then, interchanging the indices $a$ and $b$ in the last term leads to

$^2$ Notice that $S(t)$ can be viewed as being analogous to a thermodynamic non-equilibrium potential [?], allowing the determination of the characteristic quantities $S_{ab}$ by functional derivatives. For example, if $S_{ba} = S_{ab}$, we have $S_{ab} = \frac{1}{2T} \frac{\partial^2 S}{\partial \rho_a \partial \rho_b}$. 

\[ \frac{dS(t)}{dt} = \sum_a \int dx \frac{\partial \rho_a(x,t)}{\partial t} \sum_b \left(P_{ab} + P_{ba}\right) \rho_b(x,t). \]  

(9.20)

If \( P_{ab} \) is antisymmetric (i.e. \( P_{ba} = -P_{ab} \)), \( S \) is obviously an invariant of motion. However, in the following we will focus on the case \( P_{ba} = P_{ab} \) of symmetric interactions, which applies to our pedestrian example. Inserting (9.5) and (9.13) into (9.20), and applying (9.40), we get

\[ \frac{dS(t)}{dt} = -\sum_a \int dx \frac{\partial}{\partial x} \left( \rho_a(x,t) \frac{\partial S_a(x,t)}{\partial x} \right) \left( \sum_b S_{ab} \frac{\partial \rho_b(x)}{\partial x} \right). \]  

(9.21)

Making use of partial integration (for spatially periodic systems), we finally arrive at

\[ \frac{dS(t)}{dt} = 2 \sum_a \int dx \rho_a(x,t) \left( \frac{\partial S_a(x,t)}{\partial x} \right)^2 \geq 0. \]  

(9.22)  

As the overall success \( S(t) \) is a bounded function and monotonously growing (in cases of symmetrical interactions), it will eventually evolve towards its maximum. This establishes self-optimization for symmetrical interactions and can be easily transferred to discrete or higher-dimensional spaces. In case of slightly asymmetric interactions, small non-linear contributions to (9.40), or small diffusion \( D_a \), relation (9.22) will still be a good approximation, i.e. the system will behave close to optimal. In Sec. 9.2.4, we will see that more or less symmetric interactions are very natural for the kind of self-organizing systems we are considering, here. It is worth pointing out that the above result is reminiscent of dissipation functions in thermodynamics and may be viewed as a generalization of the Onsager principle of minimum dissipation [?, ?].

According to (9.22), the finally resulting stationary solution \( \rho_a(x) \) is characterized by

\[ \rho_a(x) = 0 \quad \text{or} \quad \frac{\partial P_a(x)}{\partial x} = \sum_b S_{ab} \frac{\partial \rho_b(x)}{\partial x} = 0 \]  

(9.23)

for all \( a \), which is fulfilled by homogeneous or by step-wise constant solutions. Hence, the stationary solution can be non-homogeneous, which is quite interesting.

### 9.2.3 Generalization to Other Systems

Although we will focus on motion in “real” space like the motion of pedestrians or bacteria, our model may also be applied to changes of positions in abstract spaces, e.g. to opinion changes on an opinion scale [?, ?].

many driven systems can be represented as a game between interacting (sub-)populations, we have constructed a functional for such systems, which is related to thermodynamic non-equilibrium potentials and can be interpreted
as overall (expected) success. In cases of symmetric interactions among the populations, this function increases monotonically in the course of time, meaning that the overall success of these systems is optimized. In other words, as individual entities are trying to maximize their own success, these systems tend to reach a state with the highest global success, which is not trivial at all.

The quantities $\xi_\alpha(t)$ are random variables allowing us to consider individual variations of the success, which may be “real” or due to uncertainty in the evaluation or estimation of success.

\subsection*{9.2.4 Self-Organization}

Our generalized model allows to describe all kinds of different combinations between attractive or profitable and repulsive or competitive interactions within and among the different populations. It is, therefore, desirable to know the exact conditions under which the corresponding system forms a self-organized, i.e. a non-homogeneous state. In order to derive these, we will carry out a linear stability analysis around the homogeneous stationary solution $\rho_e^a = N_a/I$, where $I$ again denotes the spatial extension of the system. For simplicity, we will restrict ourselves to the case of two (sub-)populations $a, b \in \{1, 2\}$.

We start with the continuity equation (9.16) and linearize the partial differential equations:

$$
\frac{\partial \rho_a}{\partial t} = -\rho_a \frac{\partial^2 S_a}{\partial x^2} + D_a \frac{\partial^2 \rho_a}{\partial x^2} \\
= -\sum_{b=1}^{2} \rho_a P_{ab} \frac{\partial^2 \rho_b}{\partial x^2} + D_a \frac{\partial^2 \rho_a}{\partial x^2}.
$$

(9.24)

Inserting into these equations the ansatz

$$
\rho_a(x, t) = \rho_e^a + \tilde{\rho}_a e^{i\kappa x + \lambda t},
$$

(9.25)

where $\kappa$ has the meaning of a wave number, leads to the following linear eigenvalue problem with eigenvalue $\lambda$:

$$
\lambda \begin{pmatrix} \hat{\rho}_1 \\ \hat{\rho}_2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \hat{\rho}_1 \\ \hat{\rho}_2 \end{pmatrix}.
$$

(9.26)

Herein, we have

$$
A_{11} = \kappa^2 (\rho_e^1 P_{11} - D_1), \\
A_{12} = \kappa^2 \rho_e^1 P_{12}, \\
A_{21} = \kappa^2 \rho_e^2 P_{21}, \\
A_{22} = \kappa^2 (\rho_e^2 P_{22} - D_2).
$$

(9.27)
The linear system of equations can be solved for the two eigenvalues
\[
\lambda_{\pm} = \frac{A_{11} + A_{22}}{2} \pm \sqrt{(A_{11} + A_{22})^2 - 4(A_{11}A_{22} - A_{12}A_{21})}. 
\tag{9.28}
\]
In order for the homogeneous solution to be stable, the real values of both eigenvalues need to be negative. This requires
\[
(A_{11} + A_{22}) < 0 \quad \text{and} \quad (A_{11}A_{22} - A_{12}A_{21}) > 0. 
\tag{9.29}
\]
Inserting the above abbreviations, the homogeneous solution should be unstable under the conditions
\[
\rho_1^e P_{11} + \rho_2^e P_{22} > D_1 + D_2
\tag{9.30}
\]
or
\[
(P_{12} P_{21} - D_1)(P_{22} - D_2) < \rho_1^e \rho_2^e P_{12} P_{21}. 
\tag{9.31}
\]
If one of these conditions is fulfilled, we expect the self-organization of a non-homogeneous pattern based on the non-linear interactions in space. It is interesting to note that, in the case of spatial invariance (i.e. periodic boundary conditions), self-organization directly implies spontaneous symmetry-breaking, and we expect a pronounced history-dependence of the resulting state. Nevertheless, when averaging over a large ensemble of simulation runs with different random seeds, we again expect a homogeneous distribution, since this is the only result compatible with translation invariance.

### 9.2.5 Optimal Self-Organization

Let us first discuss the case \( p_a \approx 0 \) of negligible diffusion. For this case, the relations (9.29) imply that the homogeneous solution of the model (cf. Figure 9.8a) is unstable if
\[
\rho_1^e P_{11} + \rho_2^e P_{22} > 0
\tag{9.32} \quad \{\text{cond1}\}
\]
or
\[
P_{12} P_{21} > P_{11} P_{22}. 
\tag{9.33} \quad \{\text{cond2}\}
\]
In other words, if one of the conditions (9.32) or (9.33) is fulfilled, the stable stationary solution is a self-organized, non-homogeneous state. As we will see in the following, different cases are possible.

Let us assume in the following that condition (9.32) is not valid. Then, we can show that self-organizing systems tend to reach an optimal state, as they tend to be symmetric. This can be seen as follows: Introducing
\[
\mathcal{T} = \frac{P_{12} + P_{21}}{2} \quad \text{and} \quad \Delta P = \frac{P_{12} - P_{21}}{2}, 
\tag{9.34}
\]
we have
and the condition (9.33) for the self-organization of a non-homogeneous solution becomes
\[(P + \Delta P)(P - \Delta P) = P^2 - (\Delta P)^2 > P_{11}P_{22}.\] (9.36)

If the interaction strengths \(|P_{11}|, |P_{22}|, |P_{12}|, |P_{21}|, \text{ and } |\bar{P}|\) have the same order of magnitude, this condition for self-organization can only be fulfilled for small \(|\Delta P|\), i.e.
\[P_{12} \approx P_{21}.\] (9.37)

Hence, there is a tight connection between self-organization and self-optimization in the considered kinds of systems: If there is self-organization, it is likely to come with optimality, at least approximately. This may be termed self-organized optimality or optimal self-organization. Lane formation of pedestrians is an example for this. However, there are also “non-optimal” forms of self-organization like in systems for which no Lyapunov function exists. A typical example are systems with oscillating or chaotic states.

### 9.3 Simulation Results for a Discrete Model

Note that, in the case of an unstable eigenvalue, it is the mode with the largest wave number (i.e. with the shortest wave length) that grows fastest. This somewhat unrealistic behavior is a consequence of simplifications made, namely the linear approximation underlying relation (9.10). In reality, the spatial extension of the entities will introduce a natural cutoff for the wave lengths. In order to reflect this, we will now set up a discrete version of the model.

For this, we subdivide the space into \(M\) cells \(m\) of width \(\Delta x\). Then, we have \(A\) subpopulations \(a\) with \(N_a\) entities distributed over \(M\) cells \(m \in \{1, \ldots, M\}\). We denote the number of entities in cell \(m\) at time \(t\) belonging to subpopulation \(a\) by \(N^a_m(t)\), i.e.
\[N_a = \sum_{m=1}^{M} N^a_m(t).\] (9.38)

Moreover, we represent the kind of interaction and the interaction strength between two entities of subpopulations \(a\) and \(b\) by the constant payoff \(P_{ab}\). Finally, we update the locations of the entities \(i\) according to the following rules:
9.3 Simulation Results for a Discrete Model

1. Select the entity \( i \) at random. If \( i \) is located in cell \( m \) and belongs to subpopulation \( a \), determine the individual success

\[
S_i(n, t) = \sum_b P_{ab} N_b^a(t) + \xi_i(t) \quad (9.40)
\]

for \( n = m \) and the nearest neighbors \( n = m \pm 1 \), where \( \xi_i(t) \) are random fluctuations uniformly distributed in the interval \([-p_a, p_a]\), so that \( p_a \) denotes the fluctuation strength.

2. Move to the neighboring cell \( m \pm 1 \) with probability

\[
P_a(m \pm 1|m; t) \propto \max\{0, S_a(m \pm 1, t) - S_a(m, t)\} \quad (9.41)\]

3. Repeat steps 1 and 2 until the locations of \( N = \sum_a N_a \) entities were updated.

4. Return to step 1.

EXERCISE: SHOW THAT A SECOND-ORDER TAYLOR APPROXIMATION LEADS AGAIN TO EQUATION (9.16).

Formula (9.40) calculates the expected effect of interactions with other entities. \( S_a(m, t) \) is a potential function, see Eq. (??). In the language of game theory, it can be called the expected success, since, according to the proportional imitation rule (9.41), an entity \( i \) moves to a neighboring cell \( m \pm 1 \) only if it can increase the value of \( S_i \). The values \( P_{ab} \) may be interpreted as payoffs in interactions between two entities of subpopulations \( a \) and \( b \). \( P_{ab} \) is positive for attractive, cooperative, or profitable interactions, while it is negative for repulsive, competitive, or loss-making interactions. We applied a random sequential update rule, as it seems to be most reasonable [??]. However, a parallel update, which defines a simple cellular automaton [??], yields qualitatively the same results (even nicer looking ones).

In computer simulations of the discrete model of success-driven motion in space, we consider two subpopulations \( a \in \{1, 2\} \) and \( N_1 = N_2 = 100 \) entities \( i \) in each subpopulation, which are distributed over \( M = 20 \) cells. The payoff matrix \( (P_{ab}) \) will be represented by the vector \( \mathbf{P} = (P_{11}, P_{12}, P_{21}, P_{22}) \), where we will restrict ourselves to \( |P_{ab}| \in \{1, 2\} \) for didactical reasons. For symmetric interactions between subpopulations, we have \( P_{ab} = P_{ba} \), while for asymmetric interactions, there is \( P_{ab} \neq P_{ba} \), if \( a \neq b \). For brevity, the interactions within the same population will be called self-interactions, those between different populations cross-interactions.

To characterize the different resulting spatial patterns, one can use order parameters of the form

\[
\Theta(y) = \frac{1}{T} \sum_{i=1}^{T} (y_i - \overline{y}_i)^2 \quad \text{with} \quad \overline{y}_i = \frac{1}{T} \sum_{i=1}^{T} y_i \quad (9.42)
\]

to measure the variances of
\[ y_i = \left( n_1^i / n_1^2 + n_2^i / n_2^2 \right) \] (i.e. the deviation from a homogeneous occupation of all sites), or
\[ y_i = \left( n_1^i / n_1^2 - n_2^i / n_2^2 \right) \] (i.e. the difference in the degree of occupation by different subpopulations).

\( \Theta(n^1 + n^2) \) is sensitive to (attractive or repulsive) agglomeration (i.e. to clustering with empty sites in between), and \( \Theta(n^1 + n^2) \) recognizes, when the two subpopulations tend to use different sites (as for segregation or repulsive agglomeration).

Let us first discuss the the symmetric case with \( \rho_{01} = \rho_{02} = \rho, P_{11} = P_{22} = P, P_{12} = P_{21} = Q \), and vanishing diffusion \( D_1 = D_2 = 0 \). Then, condition (9.32) reduces to \( 2\rho P > 0 \), and condition (9.33) becomes \( Q^2 > P^2 \). We can distinguish the following solutions (for representatives see the next sections):

A) If \( P < 0 \) and \( Q^2 < P^2 \) [i.e. \( P < 0 \) and \( P < Q < -P \)], a homogeneous distribution \( \rho_a(x,t) = \rho \) over all sites in both subpopulations is stable with respect to small perturbations (which corresponds to disorder).

B) If \( P < 0 \) (self-repulsion) and \( Q < 0 \) (repulsion between the subpopulations), but \( Q < P \), we should find segregation (with a tendency that all sites are equally occupied, but either by one subpopulation or by the other).

C) If \( Q < 0 \) (repulsion), but \( P > 0 \) (self-attraction), we expect repulsive agglomeration (i.e. both subpopulations should cluster at different sites, with empty sites in between).

D) If \( Q > 0 \) (attraction) and \( Q > -P \), we should have attractive agglomeration (clustering of both subpopulations at the same sites, with empty sites in between).

Consequently, on the line \( Q = (P - 1)/2 \) (i.e. for \( P = 2Q + 1 \)), we should cross the phase boundary between disorder and segregation at \( P = -1 \), the one between segregation and repulsive agglomeration at \( P = 0 \), and the one between repulsive and attractive agglomeration at \( P = +1 \). This is, in fact, confirmed by our numerical simulations (see Fig. 9.5b), so that we can trust the instability analysis based on our instability analysis.

9.3.1 Segregation

Notice that \( P_{ab} \) is assumed to be independent of the position (i.e., translation invariant), while the total payoff \( \sum_b P_{ab} n_b^i(t) \) due to interactions depends on the distribution of entities over the system. The latter is an essential point for the possibility of fluctuation-induced self-organization.

To characterize the level of self-organization in each subpopulation \( a \), we can, for example, use the overall successes

\[ S_a(t) = \frac{1}{T^2} \sum_i \sum_b n_b^i(t) P_{ab} n_b^i(t), \quad (9.43) \]
Fig. 9.5. (a) Theoretical phase diagram of the four qualitatively different patterns resulting for the symmetric model without diffusion: A = disordered phase, B = segregation, C = repulsive agglomeration, D = attractive agglomeration. (b) Order parameters along the line $Q = (P - 1)/2$ (see solid line in Fig. 12.1a), averaged over 20 runs after a time period of 20 000($N_1 + N_2$) update steps with $I = 200$ cells and $N_1 = N_2 = 2000$ entities in each of two subpopulations. The theoretically predicted phase transitions at $P = -1, P = 0$, and $P = 1$ (see black triangles in Fig. 12.1a) are clearly visible.

the variances

$$V_a(t) = \frac{1}{I^2} \sum_i [n_{i}^a(t) - n_{\text{hom}}^a]^2,$$  (9.44)

or the alternation strengths

$$A_a(t) = \frac{1}{I^2} \sum_i [n_{i}^a(t) - n_{i-1}^a(t)]^2.$$  (9.45)

### 9.3.2 Symmetric Interactions

By analogy with a more complicated model [?] it is expected that the global overall success $S(t) = \sum_a S_a(t)$ is an increasing function in time, if the fluctuation strengths $D_a$ are zero. However, what happens at finite noise amplitudes $D_a$ is not exactly known. One would usually expect that finite noise tends to obstruct or suppress self-organization, which will be investigated in the following.

We start with the payoff matrix $P = (2, -1, -1, 2)$ corresponding to positive (or attractive) self-interactions and negative (or repulsive) cross-interactions. That is, entities of the same subpopulation like each other, while entities of different subpopulations dislike each other. The result will naturally be segregation (“ghetto formation”) [? ?], if the noise amplitude is
small. However, segregation is suppressed by large fluctuations, as expected (see Fig. 9.6).

![Graph of entity distribution over cells](image)

**Fig. 9.6.** Resulting distribution of entities at $t = 4000$ for the payoff matrix $P = (2,-1,-1,2)$ at small fluctuation strength $D_a = 0.1$ (top) and large fluctuations strength $D_a = 5$ (bottom).

However, for medium noise amplitudes $D_a$, we find a much more pronounced self-organization (segregation) than for small ones (compare Fig. 9.7 with Fig. 9.6). The effect is systematic insofar as the degree of segregation (and, hence, the overall success) increases with increasing noise amplitude, until segregation breaks down above a certain critical noise level.

Let us investigate some other cases: For the structurally similar payoff matrix $(1,-2,-2,1)$, we find segregation as well, which is not surprising. In contrast, we find agglomeration for the payoff matrices $(1,2,2,1)$ and $(2,1,1,2)$. This agrees with intuition, since all entities like each other in these cases, which makes them move to the same places, like in the formation of settlements [?], the development of trail systems [?, ?, ?], or the build up of slime molds [?, ?]. More interesting is the case corresponding to the payoff matrix $(-1,2,2,-1)$, where the cross-interactions are positive (attractive), while the self-interactions are negative (repulsive), see Fig. 9.8. One may think that this
causes the entities of the same subpopulation to spread homogeneously over the system, and in all cells would result an equal number of entities of both subpopulations, which is compatible with mutual attraction. However, this homogeneous distribution turns out to be unstable with respect to fluctuations. Instead, we find agglomeration! Corresponding simulation results for different noise strengths are depicted in Fig. 9.8. Again, we find that the self-organized pattern is destroyed by strong fluctuations in favour of a more or less homogeneous distribution, while medium noise strengths further self-organization.

For the payoff matrices \((-2, 1, 1, -2)\) and \((-2, -1, -1, -2)\), i.e. cases of strong negative self-interactions, we find a more or less homogeneous distribution of entities in both subpopulations, irrespective of the noise amplitude. In
Fig. 9.8. As Fig. 9.6, but for the payoff matrix $P = (-1, 2, 2, -1)$ and $D_a = 0.05$ (top), $D_a = 1.5$ (middle), and $D_a = 5$ (bottom).
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Fig. 9.9. As Fig. 9.6, but for the payoff matrix $P = (-1, -2, -2, -1)$ and $D_a = 0.05$ (top), $D_a = 0.5$ (middle), and $D_a = 5$ (bottom).

contrast, the payoff matrix $(-1, -2, -2, -1)$ corresponding to negative self-interactions but even stronger negative cross-interactions, leads to another self-organized pattern. We may describe it as the formation of lanes, as it is observed in pedestrian counterflows [?, ?] or in sheared granular media with different kinds of grains [?]. While both subpopulations tend to separate from each other, at the same time they tend to spread over all the available space (see Fig. 9.9), in contrast to the situation depicted in Figs. 9.6 and 9.7. Astonishingly enough, a medium level of noise again supports self-organized ordering, since it helps the subpopulations to separate from each other.
Fig. 9.10. Resulting distribution of entities at $t = 4000$ with saturation level $N_{\text{max}} = 50$. Top: $P = (2, -1, -1, 2)$ and $D_a = 3$. Bottom: $P = (-1, 2, 2, -1)$ and $D_a = 1.5$.

We finally mention that a finite saturation level suppresses self-organization in a surprisingly strong way, as is shown in Fig. 9.10. Instead of pronounced segregation, we will find a result similar to lane formation, and even strong agglomeration will be replaced by an almost homogeneous distribution.

### 9.3.3 Noise-Induced Ordering

A possible interpretation for noise-induced ordering would be that fluctuations allow the system to leave local minima (corresponding to partial agglomeration or segregation only). This could trigger a transition to a more stable state with more pronounced ordering. However, although this interpretation is consistent with a related example discussed in Ref. [?], the idea of a step-wise coarsening process is not supported by the temporal evolution of the distribution of entities (see Fig. 9.11) and the time-dependence of the overall success within the subpopulations (see Fig. 9.12). This idea is anyway not applicable to segregation, since, in the one-dimensional case, the repulsive clusters of different subpopulations cannot simply pass each other in order to join others of the same subpopulation.

According to Figs. 9.11 and 9.12, segregation and agglomeration rather take place in three phases: First, there is a certain time interval, during which
the distribution of entities remains more or less homogeneous. Second, there is a short period of rapid self-organization. Third, there is a continuing period, during which the distribution and overall success do not change anymore. The latter is a consequence of the short-range interactions within our model, which are limited to the nearest neighbors. Therefore, the segregation or aggregation process practically stops, after separate peaks have evolved. This is not the case for lane formation, where the entities redistribute, but all cells remain occupied, so that we have ongoing interactions. This is reflected in the non-stationarity of the lanes and by the oscillations of the overall success.

**Fig. 9.11.** Temporal evolution of the distribution of entities within subpopulation $a = 2$ for $P = (2, -1, -1, 2)$ and $D_a = 3$ (top), $P = (-1, 2, 2, -1)$ and $D_a = 1.5$ (middle), and $P = (-1, -2, -2, -1)$ and $D_a = 0.5$ (bottom).
takes place. After this “phase of preparation”, order develops in the second phase similar to crystallization, and it persists in the third phase. The role of fluctuations seems to be the following: An increased noise level avoids a rash local self-organization by keeping up a quasi-continuous distribution of entities, which is required for a redistribution of entities over larger distances. In this way, a higher noise level increases the effective interaction range by extending the first phase, the “interaction phase”. As a consequence, the resulting structures are more extended in space (but probably without a characteristic length scale, see Introduction).

It would be interesting to investigate, whether this mechanism has something to do with the recently discovered phenomenon of “freezing by heating”,

\begin{figure}
\centering
\includegraphics[width=\textwidth]{temporal_evolution.png}
\caption{Temporal evolution of the overall success within both subpopulations for $P = (2, -1, -1, 2)$ and $D_a = 3$ (top), $P = (-1, 2, 2, -1)$ and $D_a = 1.5$ (middle), and $P = (-1, -2, -2, -1)$ and $D_a = 0.5$ (bottom).}
\end{figure}
where a medium noise level causes a transition to a highly ordered (but energetically less stable) state, while extreme noise levels produce a disordered, homogeneous state again [?].

### 9.3.4 Asymmetric Interactions

Even more intriguing transitions than in the symmetric case can be found for asymmetric interactions between the subpopulations. Here, we will focus on the payoff matrix \((-1, 2, -2, 1)\), only. This example corresponds to the curious case, where individuals of subpopulation 1 weakly dislike each other, but strongly like individuals of the other subpopulation. In contrast, individuals of subpopulation 2 weakly like each other, but they strongly dislike the other subpopulation. The outcome is depicted in Fig. 9.13. It turns out that the situation is quite complex, as discussed in the next paragraph.

### 9.3.5 Noise-Induced Self-Organization

At small noise levels $D_a$, we will just find more or less homogeneous distributions of the entities. This is already different from the cases of agglomeration, segregation, and lane formation we have discussed before. Self-organization is also not found at higher noise amplitudes $D_a$, as long as we assume that they are the same in both subpopulations (i.e., $D_1 = D_2$). However, given that the fluctuation amplitude $D_2$ in subpopulation 2 is small, we find an agglomeration in subpopulation 2, if the noise level $D_1$ in subpopulation 1 is medium or high, so that subpopulation 1 remains homogeneously distributed. The order in subpopulation 2 breaks down, as soon as we have a relevant (but still small) noise level $D_2$ in subpopulation 2 (see Fig. 9.13).

Hence, we have a situation where asymmetric noise with $D_1 \neq D_2$ can facilitate self-organization in a system with completely homogeneous initial conditions and interaction laws, where we would not have ordering without any noise. We call this phenomenon noise-induced self-organization. It is to be distinguished from the noise-induced increase in the degree of ordering discussed above, where we have self-organization even without noise, if only the initial conditions are not fully homogeneous.

The role of the noise in subpopulation 1 seems to be the following: Despite of the attractive interaction with subpopulation 2, it suppresses an agglomeration in subpopulation 1, in particular at the places where subpopulation 2 agglomerates. Therefore, the repulsive interaction of subpopulation 2 with subpopulation 1 is effectively reduced. As a consequence, the attractive self-interaction within subpopulation 2 dominates, which gives rise to the observed agglomeration.

The temporal development of the distribution of entities and of the overall success in the subpopulations gives additional information (see Fig. 9.14). As in the case of lane formation, the overall success fluctuates strongly, because the subpopulations do not separate from each other, causing ongoing
Fig. 9.13. Distributions for $P = (-1, 2, -2, 1)$ and $D_1 = D_2 = 0.5$ (top), $D_1 = 50$, $D_2 = 0.5$ (second), $D_1 = 5000$, $D_2 = 0.5$ (third), $D_1 = 5000$, $D_2 = 5$ (bottom).
interactions. Hence, the resulting distribution is not stable, but changes continuously. It can, therefore, happen, that clusters of subpopulation 2 merge, which is associated with an increase of overall success in subpopulation 2 (see Fig. 9.14).

**Fig. 9.14.** Temporal evolution of the distribution of entities within subpopulation $a = 2$ (top) and of the overall successes (bottom) for $P = (-1, 2, -2, 1)$ and $D_1 = 50$, $D_2 = 0.5$. 

{fig9}
Stochasticity and Fluctuations

Driver behavior is, of course, far from being deterministic. Therefore, driver behavior varies in time, particularly because of errors in the estimation of distances, speeds, and relative velocities (see Sec. ??), because of changing environmental conditions and driver conditions (fatigue??), and because of imperfect vehicle control. These factors are sometimes accounted for by fluctuation (or “noise”) terms.

10.0.6 The Discretized Optimal Velocity Model

MAKE THIS AN EXERCISE: CALCULATION OF THE CHARACTERISTIC CONSTANTS?

To construct a mathematical link between the Nagel-Schreckenberg model and the optimal velocity model, let us discretize the latter:

\[ v_\alpha(t + \Delta t) = v_\alpha(t) + [v'_e(d_\alpha(t)) - v_\alpha(t)] \Delta t/\tau. \]  

(10.1)

Dropping the vehicle index \( \alpha \) and scaling the time by \( \Delta t = 1 \text{ s} \), distances by the cell length \( \Delta x \) (to be specified later on), and velocities by \( \Delta x/\Delta t \) results in

\[ \hat{v}_{i+1} = \hat{v}_i + \hat{\lambda}[\hat{v}'_e(\hat{d}_i) - \hat{v}_i] \],

(10.2)

where \( \hat{\lambda} = \Delta t/\tau \). In order to have integer-valued velocities \( \hat{v} \) and locations, we need to introduce a tabular function \( h(z) \). Additionally, we will add some noise for comparison with the Nagel-Schreckenberg model. The finally resulting discrete and noisy optimal velocity model reads (Helbing and Schreckenberg, 1999):

\[ \hat{v}_{i+1} = \max \left(0, \hat{v}_i + h(\hat{\lambda}[\hat{v}'_e(\hat{d}_i) - \hat{v}_i]) - \xi^{(p)}_i\right) \].

(10.3)

For the floor function \( h(z) = \lfloor z \rfloor \), the argument \( z \) is rounded down to the largest natural number \( n \leq z \). Then, the above equation implies
\[ \dot{v}_{i+1} \leq \hat{\lambda} \hat{v}'_i(d_i) + (1 - \hat{\lambda}) \dot{v}_i. \] (10.4)

This model can be specified in a way that vehicles will normally not jump into the same cell or over each other. However, like in the continuous optimal velocity model itself, the discrete version is sensitive to the choice of the parameter \( \hat{\lambda} = \Delta t/\tau \) and the integer-valued velocity-distance function \( \hat{v}'_i(d) \). A value \( \hat{\lambda} \approx 0.77 \) seems to be optimal. For city traffic, the additional specifications \( \hat{v}'_i(d) = \min(d - 1, 3) \) and \( \Delta x = 6.25 \text{ m} \) yield realistic results. The favourite properties of this model are that

- jam formation is not affected by reducing the fluctuation strength (see Fig. ??),
- the characteristic constants of traffic are reproduced (see Secs. ?? and ??),
- the critical densities and the characteristic constants can be calculated (see Sec. ??),
- a relation with other cellular automata can be constructed.

For example, the Nagel-Schreckenberg model corresponds to the selection \( \hat{\lambda} = 1 \) and \( \hat{v}'_i(d) = \min(d - 1, \hat{v}_{\text{max}}) \). Moreover, \( h(z) = \lfloor z \rfloor, \) if \( z < 1 \), otherwise \( h(z) = 1 \), which limits acceleration, while deceleration capabilities are unlimited. Nevertheless, these two mathematically related models lead to a different dynamics (compare Figs. ?? and ??). In the discretized optimal velocity model, velocity changes \( d\hat{v}'_i(d)/d\hat{d} \) with the distance are large enough to cause a linear instability. Therefore, the model describes emergent traffic jams even in the deterministic limit \( \hat{p} \to 0 \).

### 10.1 Stochastic Differential Equations

In the onedimensional driven-many particle system we discuss, point-like particles \( i \) change their location \( x_i(t) \) in time \( t \) according to the equation of motion

\[ \frac{dx_i}{dt} = v_i(t), \] (10.5)

and their velocity change \( dv_i/dt \) is assumed to be given by the following stochastic acceleration equation:

\[ \{0\} \]

\[ \frac{dv_i}{dt} = \frac{v_0 - v_i}{\tau} + f(d_i) - \hat{\gamma} f(d_{i-1}) + \xi_i(t). \] (10.6)

Here, \( v_0 \) denotes the “free” or “desired” velocity and \( \xi_i(t) \) represents a white noise fluctuation term satisfying

\[ \langle \xi_i(t) \rangle = 0, \]

\[ \langle \xi_i(t) \xi_j(t') \rangle = D\delta_{ij}\delta(t - t'), \] (10.7)

where \( D \) is a velocity-diffusion constant. The particle mass \( m_i \) has been set to 1, and \( f(d_i) \leq 0 \) describes a repulsive interaction force, which depends on
the particle distance \( d_i(t) = x_{i+1}(t) - x_i(t) \). The term \( \hat{\gamma} f(d_{i-1}) \) with \( 0 \leq \hat{\gamma} \leq 1 \) allows to study different cases: \( \hat{\gamma} = 1 \) corresponds to the classical case of symmetrical interactions in forward and backward direction fulfilling the physical law of “actio = reactio”, while \( \hat{\gamma} = 0 \) corresponds to the case of only forwardly directed interactions as, for example, applicable to vehicles.

In order to determine the statistical distributions of particle velocities and distances, it is helpful to rewrite the above stochastic differential equation (Langevin equation) in terms of an equivalent Fokker-Planck equation. With the definitions

\[
d = (d_1, \ldots, d_n), \quad v = (v_1, \ldots, v_n),
\]

\[
W(d_i, d_{i-1}) = v_0 + \tau [f(d_i) - \hat{\gamma} f(d_{i-1})],
\]

and \( P(d_1, \ldots, d_n, v_1, \ldots, v_n, t) = P(d, v, t) \),

this Fokker-Planck equation reads

\[
\frac{\partial P}{\partial t} = \sum_{i=1}^{n} \left\{ -\frac{\partial}{\partial d_i} \left[ (v_{i+1} - v_i) P \right] \right\}_{=dd_i/\partial t} - \frac{\partial}{\partial v_i} \left[ \left( \frac{W(d_i, d_{i-1}) - v_i}{\tau} \right) P \right]_{=d_v_i/\partial t} + \frac{D}{2} \frac{\partial^2 P}{\partial v_i^2},
\]

where we assume periodic boundary conditions \( v_{k+n}(t) = v_k(t) \) and \( d_{k+n}(t) = d_k(t) \) for a onedimensional ring of length \( L \). In the following, we will show that the ansatz

\[
P(d, v, t) = P(d_1, \ldots, d_n, v_1, \ldots, v_n)
= \mathcal{N}(t) e^{-\sum_i[U/d_i]/\hat{\theta}(t) + B d_i} e^{-\sum_j[v_j - V(t)]^2/\tau \hat{\theta}(t)}
\]

is a stationary solution of the above Fokker-Planck equation, if the parameters \( V \) and \( \theta \) are properly chosen. In Eq. (10.11),

\[
U(d_i) = \frac{1 + \hat{\gamma}}{2} \int_0^{d_i} df
\]

is called the interaction potential,

\[
\mathcal{N}(t) = \left[ \int d^n s \int d^n v P(d, v, t) \right]^{-1}
\]

is the normalization constant, and the parameter \( B \) determines the actual particle density (which is required to ensure \( \sum_i d_i = L \)). Moreover,

\[
V(t) = \langle v_i \rangle = \int d^n s \int d^n v v_i P(d, v, t)
\]
is the average particle velocity and
\[
\theta(t) = \langle (v_i - V)^2 \rangle = \int d^n s \int d^n v (v_i - V)^2 P(d, v, t) \tag{10.15}
\]
the velocity variance. In the following, we will restrict our investigation to the stationary case with \(dV/dt = 0\) and \(d\theta/dt = 0\), which presupposes that the deterministic part of Eq. (10.49) fulfils the linear stability condition
\[
(1 - \hat{\gamma})^2 \frac{df(L/n)}{ds} < \frac{1 + \hat{\gamma}}{2\tau^2}, \text{CHECK!} \tag{10.16}
\]
Otherwise, dynamic patterns such as stop-and-go waves may emerge from the dissipative interactions of driven particles. Notice that in the Hamiltonian case, \(\hat{\gamma} = 1\), the stability condition is always satisfied. Furthermore, the factorization assumption (10.11) implies that the variables \(d_i\) and \(v_{i+1}\) are statistically independent. According to numerical simulations, this is only the case if \((1 - \hat{\gamma})^2 df/ds\) is much smaller than the right-hand side of (10.16).

Differentiation of Eq. (10.11) gives
\[
- \sum_i \frac{\partial}{\partial d_i} (v_{i+1} - v_i) P = \sum_i (v_{i+1} - v_i) \left[ \frac{1}{\theta} \frac{dU(d_i)}{dd_i} + B \right] P
= \sum_i (v_{i+1} - v_i) \left[ \frac{(1 + \hat{\gamma}) f(d_i)}{2\theta} + B \right] P \tag{10.17}
\]
and
\[
- \sum_i \frac{\partial}{\partial v_i} \left( \frac{W(d_i, d_{i-1}) - v_i}{\tau} \right) P = \sum_i P - \sum_i \frac{W(d_i, d_{i-1}) - v_i}{\tau} \left[ - \frac{(v_i - V)}{\theta} \right] P, \tag{10.18}
\]
and
\[
\sum \frac{D}{2} \frac{\partial^2 P}{\partial v_i^2} = \frac{D}{2} \sum_i \left[ \frac{1}{\theta} + \left( - \frac{v_i - V}{\theta} \right)^2 \right] P. \tag{10.19}
\]
We will now insert this into Eq. (10.10) and use the fact that
\[
\sum_i g_{i+1} P = \sum_i g_i P \tag{10.20}
\]
for any \(i\)-dependent variable \(g_i\), i.e. indices can be shifted because of the assumed periodic boundary conditions. In this way we find
\[
\frac{\partial P}{\partial t} = \frac{1}{\theta} \sum_i (v_{i+1} - v_i) f(d_i) P + \sum_i \frac{P}{\tau} - \sum_i \frac{DP}{2\theta}
+ \frac{1}{\theta} \sum_i \left[ \frac{v_0 - v_i}{\tau} + f(d_i) - \hat{\gamma} f(d_{i-1}) \right] (v_i - V) P + \frac{D}{2\theta^2} \sum_i (v_i - V) P \tag{10.21}
\]
Remarkably, this equation does not depend on the Lagrange parameter $B$ anymore, which is needed to adjust to the particle density.

Note that ansatz (10.11) can only be a stationary solution with $\partial P/\partial t = 0$, if

$$\frac{1}{\bar{\theta}} = \frac{2}{D \tau}. \quad (10.22)$$

This relationship corresponds to the fluctuation-dissipation theorem. Applying it also to the last term of Eq. (10.21) and using the decompositions

$$(v_{i+1} - v_i) = (v_{i+1} - V) - (v_i - V) \quad \text{and} \quad (v_0 - v_i) = (v_0 - V) - (v_i - V),$$

we find

$$\frac{\partial P}{\partial t} = \frac{1 - \hat{\gamma}}{2\theta} \left[ \sum_i (v_{i+1} - V) f(d_i) P + \sum_i (v_i - V) f(d_i) P \right] + \frac{1}{\bar{\theta}} \sum_i \frac{(v_0 - V)(v_i - V)}{\tau} P. \quad (10.23)$$

We will distinguish the following cases:

1. In the case of a classical many-particle system with momentum conservation ($\hat{\gamma} = 1$) and energy conservation, i.e. no dissipation ($\tau \to \infty$), we find $\partial P/\partial t = 0$, i.e. ansatz (10.11) is an exact stationary solution of the Fokker-Planck equation (10.10).

2. In the case of forwardly directed interactions as in vehicle traffic ($\hat{\gamma} = 0$), we have to show that the additional term

$$\frac{1}{\bar{\theta}} \sum_i (v_{i+1} - V) \left[ f(d_i) + \frac{v_0 - V}{\tau} \right] P \quad (10.24)$$

disappears (where we have again shifted indices). Let us first note that, with the factorization assumption (10.11), i.e. with the statistical independence of $d_i$ and $v_{i+1}$, we have

$$\lim_{n \to \infty} \frac{1}{n} \sum_i (v_{i+1} - V) \left[ f(d_i) + \frac{v_0 - V}{\tau} \right] = \left[ \lim_{n \to \infty} \frac{1}{n} \sum_i (v_{i+1} - V) \right] \left[ \lim_{n \to \infty} \frac{1}{n} \sum_i f(d_i) + \frac{v_0 - V}{\tau} \right]. \quad (10.25)$$

The first factor vanishes because of $V = \lim_{n \to \infty} \frac{1}{n} \sum_i v_i$, but the second factor disappears as well: Dividing Eq. (10.49) by $n$ and summing up over $i$ gives

$$\frac{1}{n} \sum_i \frac{dv_i}{dt} = \frac{1}{n} \sum_i \frac{v_0 - v_i}{\tau} + \frac{1}{n} \sum_i f(d_i) + \frac{1}{n} \sum_i \xi_i(t). \quad (10.26)$$

In the limit $n \to \infty$ of large enough particle numbers $n$, the left-hand side converges to $dV/dt$, while the last term on the right-hand side converges to 0. In the assumed stationary case with $dV/dt = 0$ and using $v_0 - v_i = (v_0 - V) - (v_i - V)$, this implies
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\[ 0 = \lim_{n \to \infty} \frac{1}{n} \sum_{i} \left[ \frac{v_0 - v_i}{\tau} + f(d_i) \right] = \lim_{n \to \infty} \frac{1}{n} \sum_{i} \left[ \frac{v_0 - V}{\tau} + f(d_i) \right] \] (10.27) \{because\}

because of \( \frac{1}{n} \sum_{i} v_i = V = \frac{1}{n} \sum_{i} V \).

Note that Eq. (10.23) can be shown for any many-particle distribution of the form

\[ P(d_1, \ldots, d_n, v_1, \ldots, v_n, t) = \prod_{i=1}^{n} g(d_i, t) \prod_{j=1}^{n} h(v_j, t), \] (10.28)

which is composed of single-particle distribution functions \( P_1(d_i, v_i, t) = g(d_i, t)h(v_i, t) \). Beyond Eq. (10.23), one can derive

\[ \frac{1}{n} \sum_{i} \frac{d}{dt} [g(d_i, t)h(v_{i+1}, t)] = \frac{1}{n\theta} \sum_{i} \left( v_{i+1} - V \right) \left[ f(d_i) + \frac{v_0 - V}{\tau} \right] g(d_i, t)h(v_{i+1}, t), \] (10.29)

which transfers our conclusions from the \( n \)-particle distribution function \( P \) to the average of one-particle distribution functions. However, numerical simulations show that, for small enough values of \( \tau \) and large enough particle numbers \( n \),

\[ g(d) = Ae^{-[U(d)/\theta+ Bd]} \] (10.30)

is a good approximation of the empirical distance distribution and

\[ h(v) = \frac{1}{\sqrt{2\pi\theta}} e^{-(v-V)^2/(2\theta)} \] (10.31)

a good approximation of the empirical velocity distribution. Here, \( A \) is a normalization constant, \( V \) the average velocity, and \( \theta \) the velocity variance. They are all dependent on the particle density \( \rho \).

Finally, let us investigate the Hamiltonian

\[ H = T + V = \sum_{i} \left( \frac{v_i - V}{2} + U(d_i) \right) \] (10.32)

If \( dV/dt = 0 \), we can derive the following relations:
\[
\frac{d\mathcal{H}}{dt} = \frac{dT}{dt} + \frac{dV}{dt} \\
= \sum_i (v_i - V) \frac{dv_i}{dt} + \sum_i \frac{dU(d_i)}{dt} \left( \frac{dd_i}{dx_i} \frac{dx_i}{dt} + \frac{dd_i}{dx_{i+1}} \frac{dx_{i+1}}{dt} \right) \\
= \sum_i (v_i - V) \frac{dv_i}{dt} + \sum_i \frac{1 + \gamma}{2} f(d_i)(v_{i+1} - v_i) \\
= \sum_i (v_i - V) \left( \frac{v_0 - v_i}{\tau} + f(d_i) - \hat{\gamma}f(d_{i-1}) + \xi_i(t) \right) + \sum_i \frac{1 + \gamma}{2} f(d_i)(v_{i+1} - v_i) \\
= \sum_i \frac{1 - \gamma}{2} (v_{i+1} - V)f(d_i) + \sum_i \frac{1 - \gamma}{2} (v_i - V)f(d_i) \\
+ \sum_i \left( \frac{v_0 - V}{\tau} - \sum_i (v_i - V)^2 + \sum_i (v_i - V)\xi_i(t) \right). \tag{10.33}
\]

Comparing this with (10.23) shows that

\[
\frac{\partial P}{\partial t} = P \frac{d\mathcal{H}}{dt} + \frac{1}{\theta} \sum_i \left[ \frac{(v_i - V)^2}{\tau} - (v_i - V)\xi_i(t) \right] P. \tag{10.34}
\]

Correspondingly, in the stationary state \(\partial P/\partial t = 0\) we have

\[
\frac{d\mathcal{H}}{dt} = \sum_i \left[ (v_i - V)\xi_i(t) - \frac{(v_i - V)^2}{\tau} \right] = \sum_i (v_i - V) \left( \xi_i(t) - \frac{v_i - V}{\tau} \right). \tag{10.35} \]

We will again distinguish two different cases:

1. In a conservative system with no fluctuations \((\xi_i(t) = 0 = D)\) and no dissipation \((\tau \to \infty)\), we have \(d\mathcal{H}/dt = 0\), independently of whether the interactions are symmetric or forwardly directed.

2. For many-particle systems with fluctuation terms and/or dissipation, one can show

\[
\langle \xi_i(v_i - V) \rangle = \left\langle \frac{1}{2} \frac{d(v_i - V)^2}{dt} \right\rangle - \frac{v_0 - V}{\tau} \langle v_i - V \rangle \\
+ \frac{1}{\tau} \langle (v_i - V)^2 \rangle - \langle f(d_i) - \hat{\gamma}f(d_{i-1})(v_i - V) \rangle \\
= \frac{1}{2} \frac{d\theta}{dt} - \frac{v_0 - V}{\tau} \langle v_i \rangle - \frac{\theta}{\tau} - \langle f(d_i) - \hat{\gamma}f(d_{i-1}) \rangle \langle v_i \rangle. \tag{10.36}
\]

This can be found by multiplication of Eq. (10.49) with \((v_i - V)\) and calculation of the ensemble average, using the factorization ansatz (10.11).

The first term on the right-hand side vanishes under the assumption of a stationary state. The second and the fourth term vanish because of \(\langle v_i \rangle = V\). Therefore,

\[
\langle \xi_i(v_i - V) \rangle = \frac{\theta}{\tau}. \tag{10.37}
\]
and
\[ \frac{1}{n} \sum_i \xi_i(v_i - V) \approx \frac{1}{n} \sum_i \frac{(v_i - V)^2}{\tau}, \quad (10.38) \]
if \( n \) is large enough. Without dissipation (\( \tau \to \infty \)), \( \langle \xi_i(v_i - V) \rangle \) becomes zero, while it is finite otherwise. In any case, together with Eq. (10.35), we arrive at
\[ \langle \frac{dH}{dt} \rangle = n \left( \frac{\theta}{\tau} - \frac{1}{n} \sum_i \frac{(v_i - V)^2}{\tau} \right) = 0. \quad (10.39) \]
That is, in the statistical average we have \( dH/dt = 0 \). The same is expected for systems with many particles. As a consequence, stationary driven dissipative systems behave approximately Hamiltonian, even if the interactions are forwardly directed and Newton’s law “actio = reactio” is violated. This is why the Hamiltonian statistics
\[ P(d, v) = N e^{-\mathcal{H}/\theta} \quad (10.40) \]
(the canonical distribution) is an approximate stationary solution of our driven dissipative many-particle system (IF \( \tau \) IS SMALL ENOUGH, I.E. THE STATIONARY DISTRIBUTION IS STABLE WITH RESPECT TO FLUCTUATIONS) [Note that the contribution \( \sum_i B d_i = BL \) in Eq. (??) gives just a constant prefactor and can be absorbed into the normalization factor.] For the validity of this approximative solution (10.40), it is essential that the number \( n \) of particles is large enough and that the system is far enough away from a dynamic instability, i.e. the relaxation time \( \tau \) is small enough.

In conclusion, the equilibrium solution (10.11) of conservative many-particle systems is also a good approximation for the steady-state solutions \( \partial P/\partial t \) of driven many-particle systems of kind (10.49) with asymmetrical interactions, driving and dissipation effects, if the system is large enough, i.e. \( n \gg 1 \). For small systems, however, we expect that fluctuations become essential.

### 10.1.1 Determination of the Interaction Potential

From an experimental point of view, the theoretical distance distribution (??) is useful for determining the interaction potential \( U(d_i) \) in the presence of fluctuations \( \xi_i(t) \). Equation (??) implies
\[ \ln g(d_i) = \ln A(\rho) - U(d_i)/\theta(\rho) - B(\rho)d_i. \quad (10.41) \]
As the potential \( U(d_i) \) and its slope \( dU(d_i)/dd_i \) is assumed to disappear in the limit \( d_i \to \infty \) of large distances, the term \(-B(\rho)d_i \) dominates for large values of \( d_i \). This allows one to determine the appropriate, density-dependent
value of $B$ (see Fig. ??), while the potential $U(d_i)$ itself should be density-independent. Finally, $A(\rho)$ can be numerically calculated as normalization constant of the distance distribution $g(d_i)$.

WIRD DIESER ABSCHNITT UEBERHAUPT GEBRAUCHT? For the adjustment of the value of $B$, one may define two integrals

$$Z(B) = \int_0^\infty dd_i \ e^{-\left[U(d_i)/\theta + Bd_i\right]}$$  \hspace{1cm} (10.42)$$

and

$$J(B) = \int_0^\infty dd_i \ d_i e^{-\left[U(d_i)/\theta + Bd_i\right]} = -\frac{dZ(B)}{dB}.$$  \hspace{1cm} (10.43)$$

The choice

$$A(B) = \frac{1}{Z(B)}$$  \hspace{1cm} (10.44)$$

obviously guarantees normalization of the distance distribution function $g(d_i)$, while

$$\frac{1}{\rho} = A(B)J(B) = -\frac{dZ(B)}{dB} = -\frac{d\ln Z(B)}{dB}$$  \hspace{1cm} (10.45)$$

determines the particle density $\rho(B)$ belonging to the respective value of $B$. Conversely, the right value of $B(\rho)$ for a given particle density $\rho$ is given by the inverse function of $\rho(B)$, which also implies a density-dependence of the normalization constant $A(B(\rho))$. This concludes the procedure of obtaining the parameters and functions in the distance distribution function $g(d_i)$.

Figure ?? shows the estimated values of $U(d_i)$, $A(\rho)$, and $B(\rho)$ as determined from simulated particle data for the assumed interaction potential

$$U(d_i) = \ldots \ldots \ldots \ldots \ldots \ldots \ldots$$  \hspace{1cm} (10.46)$$

It turns out that the numerically reconstructed functions (symbols) are well compatible with the original ones (solid lines).

10.1.2 Summary and Comparison of Driven with Conservative Systems

large $\tau$: no factorization, correlations; fluctuation-dissipation theorem to be generalized; bimodal velocity distribution due to stop-and-go waves etc.

applications to queuing systems with interactions .......... 

In the first moment, it appears to be quite surprising that dissipative driven many-particle systems violating the law “actio = reactio” approximately show a Hamiltonian statistics. On top of this, the distance distribution functions are identical for symmetrical and only forwardly directed interactions. What is the reason for this? For conservative systems, the average velocity $V$ does
not change in time. The additional relaxation term \((v_0 - v_i)/\tau\) causes the average velocity \(V\) to relax to the “free” or “desired” velocity \(v_0\). However, if the law of “actio = reactio” is violated, the stationary value of the average velocity is rather given by

\[
V = v_0 + \tau f(L/n) .
\]  

(10.47)

In other words, on average the relaxation term \((v_0 - v_i)/\tau\) will produce a non-vanishing contribution

\[
\frac{v_0 - V}{\tau} = -\tau f(L/n) ,
\]  

(10.48)

which is greater than zero for repulsive forces \(f(d_i) < 0\). Consequently, the relaxation term acts similar to an additional force from the back—as in the conservative system! This aspect can be better seen by rewriting Eq. (10.49) according to

\[
\frac{dv_i}{dt} = V_*(d_i, d_{i-1}) - v_i \tau + \frac{1 + \hat{\gamma}}{2} [f(d_i) - f(d_{i-1})] + \xi_i(t) ,
\]  

(10.49)

where we have defined

\[
V_*(d_i, d_{i-1}) = v_0 + \tau \left(1 - \frac{\hat{\gamma}}{2}\right) [f(d_i) + f(d_{i-1})] .
\]  

(10.50)

For the stationary velocity we have the simple relationship

\[
V(L/n) = V_*(L/n, L/n) = v_0 + \tau (1 - \hat{\gamma}) f(L/n) .
\]  

(10.51)

Therefore, only in the conservative case \(\hat{\gamma} = 1\), we find \(V(L/n) = v_0\), as claimed above.

Despite of the above mentioned, astonishing similarities close to the steady state, there are some important differences between conservative systems and dissipative systems with forwardly directed interactions concerning their dynamics:

1. The average velocity \(V\) is \(v_0\) in the conservative case, while it is smaller in the dissipative case. There, the average velocity goes down with increasing particle density, if the interaction force is monotonously falling.
2. The conservative many-particle system always behaves dynamically stable, while the dissipative system potentially produces stop-and-go waves [see Eq. ??].
3. While close to the steady state, the fluctuation-dissipation theorem (??) is always valid for the conservative system, it requires corrections to the relaxation time close to the instability point in the dissipative case.

According to computer simulations, close to the instability point, the driven dissipative many-particle system tends to produce correlations between distances and velocities and between successive particles. Beyond the transition
point, one can even find bimodal velocity and distance distributions rather than unimodal ones. This corresponds to pattern formation phenomena that would not occur in conservative systems. Such kinds of pattern formation phenomena have, for example, been investigated by ............... The behavior of the above Fokker-Planck equation close to the instability point and beyond it will be subject of our future studies.

10.2 Noise-Induced Transitions

DISCUSS THIS IN THE SECTION NON-LINEAR INSTABILITY?
11.1 Section-Based Model for Network Flows

In the following, when modeling traffic flows in street networks, we will go back to the simple Lighthill-Whitham-Richard model because of its easier analytical tractability and because the traffic dynamics in urban environments is largely determined by traffic lights. Our network traffic model will be based on the following assumptions:

- The road network can be decomposed into road sections $j$ of homogeneous capacity (links) and nodes (without storage capacity) describing their connections.
- The parameters of vehicles such as the maximum speed $V^0_j$ and the safe time headway $T$ are assumed to be identical in the same road section $j$.
- The vehicles exit a road section in the order of entering (FIFO principle). That is, overtaking is assumed to be negligible.
- The fundamental diagram can be approximated by a triangular shape, with an increasing slope $V^0_j$ at low densities and a decreasing slope $c_0$ in the congested regime. This implies two constant characteristic speeds: $V^0_j$ corresponds to the free speed or speed limit on road section $j$, while

$$-c_0 = -\frac{1}{\rho_{\text{jam}} T}$$ (11.1)

is the dissolution speed of the downstream front of a traffic jam and the velocity of upstream propagation of perturbations in congested traffic. While $\rho_{\text{jam}}$ denotes the maximum vehicle density in vehicle queues, $T \approx 1.8s$ is the safe time gap between two successive vehicles.
- The vehicle density in traffic jams $\rho_{\text{jam}}$ is basically constant.

The particular advantage of assuming a triangular fundamental diagram is the fact that the density, average velocity, and flow inside of a road section of homogeneous flow capacity can be easily determined from the flows at
the upstream and downstream end of the road section \( j \). Therefore, we have to find coupled equations for the arrival flows \( A_j(t) \) and the departure flows (outflows) \( O_j(t) \) of the road section. The arrival flow denotes the actual inflow of vehicles into the upstream end of road section \( j \), while the departure flow is the flow of vehicles leaving road section \( j \) at its downstream end. The peak of the fundamental diagram, i.e. the quantity

\[
\hat{Q}_j = \left( T + \frac{1}{V_j^0 \rho_{jam}} \right)^{-1} = \frac{\rho_{max}}{1/c_0 + 1/V_j^0}, \tag{11.2}
\]

represents the maximum in- or outflow of road section \( j \), see Eq. (??). All the above quantities refer to flows per lane. \( I_j \) is the number of lanes and \( L_j \) the length of road section \( j \).

Later on we will learn that congested (queued) traffic always forms upstream of a bottleneck, i.e. a section of lower inflow capacity. This implies that congestion starts to form at the downstream end of a road section (link) \( j \). While the downstream front stays at the end of link \( j \), congestion grows or shrinks in upstream direction. If \( l_j(t) \) denotes the extension of congested traffic at time \( t \) measured from the downstream end of link \( j \), the length of queued traffic changes in time according to the shock wave equation (??):

\[
\frac{dl_j}{dt} = -\frac{A_j(t - \lfloor L_j - l_j(t)/V_j^0 \rfloor/V_j^0) - O_j(t - l_j(t)/c_0)}{\rho_1(t) - \rho_2(t)} . \tag{11.3}
\]

Herein,

\[
\rho_1(t) = A_j(t - \lfloor L_j - l_j(t)/V_j^0 \rfloor/V_j^0) \tag{11.4}
\]

denotes the free flow density immediately before the upstream shock front and

\[
\rho_2(t) = \lfloor 1 - TO_j(t - l_j(t)/c_0) \rfloor \rho_{jam} \tag{11.5}
\]

is the density of congested flow immediately downstream of it. This is, because traffic flow is free upstream of the shock front, and congested downstream of it.

Whenever traffic flow on link \( j \) is congested, there is a number \( \Delta N_j(t) \) is the number of stopped or delayed vehicles. This number is determined as the difference between the number

\[
N_j^\text{exp}(t) = \int_{-\infty}^{t} dt' \ A_j(t' - L_j/V_j^0) \tag{11.6}
\]

of vehicles that could have reached the downstream end of road section \( j \) and the number

\[
N_j^\text{dep}(t) = \int_{-\infty}^{t} dt' \ O_j(t') \tag{11.7}
\]
of vehicles that have actually left the road section.

\[ T_j = \frac{L_j}{V_{j0}^0} \]  \hspace{1cm} (11.8)

is the travel time on road section \( j \) under free flow conditions (i.e. at speed \( V_{j0}^0 \)). Altogether, we obtain the differential equation

\[ \frac{d\Delta N_j(t)}{dt} = \frac{dN_j^{exp}(t)}{dt} - \frac{dN_j^{dep}(t)}{dt} = A_j(t - L_j/V_{j0}^0) - O_j(t). \]  \hspace{1cm} (11.9)

The temporal change \( dN_j/t \) in the total number \( N_j(t) \) of vehicles on uncongested and congested parts of link \( j \) is rather given by the difference between the arrival and departure flow of vehicles, i.e.

\[ \frac{dN_j(t)}{dt} = A_j(t) - O_j(t). \]  \hspace{1cm} (11.10)

We can now formulate constraints for the actual arrival and departure flows, which are given by the potential arrival flows \( \hat{A}_j(t) \) and the potential departure flows \( \hat{O}_j(t) \), respectively. The actual arrival flow \( A_j(t) \) is limited by the maximum inflow \( \hat{Q}_j \), if road section \( j \) is not fully congested \( (l_j(t) < L_j) \). Otherwise (if \( l_j(t) = L_j \)), it is limited by the actual departure flow \( O_j(t - L_j/c_0) \) a time period \( L_j/c \) before, as it requires this time period until the downstream flow value has propagated up to the upstream end of the road section by forward movement of vehicles under congested traffic conditions. This implies

\[ 0 \leq A_j(t) \leq \hat{A}_j(t) = \gamma_j(t) \begin{cases} \hat{Q}_j & \text{if } l_j(t) < L_j \\ O_j(t - L_j/c_0) & \text{if } l_j(t) = L_j. \end{cases} \]  \hspace{1cm} (11.11) \{one\}

By the additional permeability factor \( \gamma_j(t) \) we allow for the modulation of the arrival flow capacity, e.g. in order to reflect the influence of a traffic light at the entrance to section \( j \). Generally, the permeability \( \gamma_j(t) \) can assume values between 0 and 1. In case of a traffic light, \( \gamma_j(t) = 1 \) corresponds to a green light for road section \( i \), while \( \gamma_j(t) = 0 \) corresponds to a red or amber light.

Finally, the potential departure flow \( \hat{O}_j(t) \) of road section \( j \) is given by the maximum outflow \( \hat{Q}_j \) from this road section, if vehicles are queued up \( (\Delta N_j > 0) \) and waiting to leave. Otherwise (if \( \Delta N_j = 0 \)) the outflow is limited by the permeability times the arrival flow \( A_j \) a time period \( L_j/V_{j0}^0 \) before, as this is the time period that entering vehicles need to reach the end of road section \( j \) when moving freely at the speed \( V_{j0}^0 \). This gives the additional relationship

\[ 0 \leq O_j(t) \leq \hat{O}_j(t) = \begin{cases} A_j(t - L_j/V_{j0}^0) & \text{if } \Delta N_j(t) = 0 \\ \hat{Q}_j & \text{if } \Delta N_j(t) > 0. \end{cases} \]  \hspace{1cm} (11.12) \{two\}
Alternatively and shorter than Eqs. (11.11) and (11.12) one can write

\[ \hat{A}_j(t) = \gamma_j(t) \max \left[ \hat{Q}_j \Theta(L_j(t) < L_j), O_j(t - L_j/c_0) \right] \quad (11.13) \] {easy1}

and

\[ \hat{O}_j(t) = \max \left[ \hat{Q}_j \Theta(\Delta N_j > 0), A_j(t - L_j/V_j^0) \right], \quad (11.14) \] {easy2}

where the Heaviside function \( \Theta \) is 1, if the argument is “true”, otherwise \( \Theta = 0 \).

Note that the above treatment of the traffic flow in a road section requires the specification of the boundary conditions only, as we have effectively integrated up Lighthill’s and Whitham’s partial differential equation over the length of the road section. However, we have a switching between the influence of the upstream and the downstream boundary conditions. While the upstream boundary controls the free flow section, the downstream boundary controls the congested part. This makes the dynamics both, complicated and interesting. The switching between the influence of the upstream or downstream boundary condition results from the maximum functions above and implies also that traffic flows in networks are described by hybrid equations. Although the dynamics is determined by linear ordinary differential equations in all regimes, the switching between the regimes can imply highly non-linear behavior (see Sec. ??)

Complementary to the above equations, we have now to specify the constraints for the nodes, i.e. the connection, merging, diverging or intersection points of the homogeneous road sections. Let the ingoing links be denoted by the index \( j \) and the outgoing ones by \( k \). To distinguish quantities more easily when we insert concrete values 1, 2, \ldots for \( j \) and \( k \), we mark quantities of outgoing links additionally by a prime (‘).

Due to the condition of flow conservation, the arrival flow into a road section \( k \) with \( I'_k \) lanes must agree with the sum of the fractions \( T_{kj} \) of all outflows \( I_jO_j(t) \) turning into road section \( j \). Additionally, the arrival flows are limited, i.e. we have

\[ I'_k A'_k(t) = \sum_j T_{kj} I_j O_j(t) \leq I'_k A'_k(t) \quad (11.15) \] {inequal}

for all \( k \). In order to ensure flow conservation at nodes, the turning fractions \( T_{kj} \geq 0 \) are normalized:

\[ \sum_k T_{kj}(t) = 1. \quad (11.16) \]

From Eq. (11.15) follows, of course, the relationship

\[ I_j O_j(t) = \sum_k T_{jk}^{-1} I'_k A'_k(t), \quad (11.17) \] {inequal2}

where \( T_{jk}^{-1} \) are the entries of the inverse matrix of \( T = (T_{kj}) \), i.e.
11.1 Section-Based Model for Network Flows

\[ \sum_k T_{lk}T_{kj}^{-1} = \delta_{lj}. \quad (11.18) \]

This again implies

\[ \sum_k T_{kj}^{-1} = \sum_k \left( \sum_l T_{lk} \right)T_{kj}^{-1} = \sum_l \left( \sum_k T_{lk}T_{kj}^{-1} \right) = \sum_l \delta_{lj} = 1. \quad (11.19) \]

The \( T_{kj}^{-1} \) can be interpreted as the fractions of the overall arrival flow \( I_k^A(t) \) that stems from the overall departure flow \( I_j^O(t) \). In their consequence, both Eq. (11.15) and (11.17) ensure Kirchhoff’s conservation law for network nodes:

\[ \sum_k I_k^A(t) = \sum_j I_j^O(t). \quad (11.20) \]

Regarding the actually resulting flows \( A_j(t) \) and \( O_j(t) \), we will assume a maximum flow principle. That is, we assume to find the maximum flows that are consistent with the constraints (....), (...) and with principle of flow conservation. For example, if one road section \( j \) feeds another road section \( k \) of different capacity, we have

\[ I_j^O(t) = I_k^A(t) = \min[I_j^O(t), I_k^A(t)], \quad (11.21) \]

e.g. an equality rather just the inequalities implied by the constraints (11.13) and (11.14). For concrete specifications of boundary conditions in cases of merges, diverges and intersections, see Ref. ??.

The above model is probably the simplest macroscopic model for traffic (or material) flows in networks that contains all the following relevant features:

1. actual and maximum flow and storage capacities of links,
2. transitions from free to congested flows due to a lack of flow or storage capacity,
3. the consideration of the propagation speeds of both, vehicles and perturbations in congested flow (particularly the upstream propagation of the downstream jam front when a traffic light turns green),
4. spillover effects (i.e. obstructions when entering fully congested road sections) and, implicitly,
5. load-dependent travel times.

The above model is general enough to include the following special cases (see Sec. ??):

1. electrical power grids and their blackouts in cases of link failures,
2. cascade failures in scenarios of disaster spreading,
3. congestion spreading in traffic and logistic systems.

A simplified version of this model can also be used to model commodity flows between different sectors of economy.
11.2 Dynamic Input-Output Model of Commodity Flows

In this section, we will make some simplifications in the previous model for network flows. We will specifically assume the following:

1. The storage capacity $I_j L_j \rho_{\text{max}}$ is not limited, i.e. $\rho_{\text{max}} \to \infty$ and $c = 0$.
2. The transport time $T_j = L_j / V_j^0$ is negligible, i.e. $T_j \approx 0$.
3. The flow capacities are the same, i.e. $I_j = 1$.

The first assumption implies $dl_j / dt = 0$ and $l_j(t) = 0$. With Eq. (11.13) this also means that

$$0 \leq A_j(t) \leq \hat{A}_j(t) = \gamma_j(t) \hat{Q}_j.$$ (11.22)

In addition, the second assumption implies

$$\frac{d\Delta N_j(t)}{dt} = \frac{dN_j(t)}{dt} = A_j(t) - O_j(t).$$ (11.23)

Assuming $N_j(t) = \Delta N_j(t) > 0$, from Eq. (11.14) follows

$$0 \leq O_j(t) \leq \hat{O}_j(t) = \hat{Q}_j.$$ (11.24)

Furthermore, Eqs. (11.23) and (11.17) result in the following system of coupled differential equations:

$$\frac{dN_j(t)}{dt} = A_j(t) - \sum_k T_{jk}^{-1} A_k(t).$$ (11.25)

Independent of the later specification of time dependence of the arrival flows $A_j(t)$, this equation is still a conservation equation for material flows. It can be considered as a variant of the dynamic input-output model of commodity flows between different economic sectors $j \in \{1, \ldots, m\}$.

When the above model of material network flows is applied as dynamic input-output model, $\Delta N_j(t)$ represents the inventory of commodities of kind $j$ (measured in units of some currency). $A_j(t)$ are the production rates of commodities of kind $j$. Considering Eq. (11.22) and the maximum flow principle formulated in Sec. 11.1, we will set

$$A_j(t) = \gamma_j(t) \hat{Q}_j = Q_j(t)$$ (11.26)

and replace $A_j(t)$ by the more common symbol $Q_j(t)$. The matrices $C = (T_{jk}^{-1})$ are called input-output matrices. Their entries $c_{jk} = T_{jk}^{-1}$ correspond to the fraction of commodities $k$ that are made up of commodities of kind $j$ (see Fig. 11.1). If we, furthermore, write up the commodity flows

$$Y_j(t) = T_{j,m+1}^{-1} A_{m+1}(t) = T_{j,m+1}^{-1} Q_{m+1}(t)$$ (11.27)

to the final consumers separately, the dynamic input-output model reads

\[ \text{For formal reasons, the consumers have been treated here like a further economic sector } j = m + 1, \text{ which does not deliver anything to other economic sectors.} \]
\[
\frac{dN_j(t)}{dt} = Q_j(t) - \sum_{k=1}^{m} c_{jk} Q_k(t) - Y_j(t). \tag{11.28}
\]

![Fig. 11.1.](image)

Main service and commodity flows among different economic sectors according to averaged input-output data of France, Germany, Japan, UK, and USA. For clarity of the network structure, we have omitted the sector ‘wholesale and retail trade’, which is basically connected with all other sectors.

As the coefficients \(c_{jk}\) reflect the composition of commodity \(k\), we have decided to insert Eq. (11.17) rather than (11.15) into Eq. (11.23). Equation (11.15) is appropriate to apply for a push system such as traffic systems, while Eq. (11.17) is appropriate for pull systems such as production systems, where the material flows are triggered by orders. The organization of (large parts of) economic systems as pull systems also means that the commodity flows or production rates \(Q_j(t)\) are subject to management and control. We will assume that the relative change \((dQ_j/dt)/Q_j\) of the production rates \(Q_j\) in time is, on the one hand, intended to adjust the inventory \(N_j(t)\) to some optimum inventory \(N^0_j\). (Too much inventory is costly, while too little implies the risks of not being able to satisfy orders. Therefore, an optimum inventory level \(N^0_j\) does exist.) On the other hand, temporal variations in the inventory level shall be avoided. Therefore, we will apply the following equation for adjustments of the production rates \(Q_j(t)\):

\[
\frac{1}{Q_j(t)} \frac{dQ_j}{dt} = \ddot{\nu}_j \left( \frac{N_j^0}{N_j(t)} - 1 \right) - \ddot{\mu}_j \frac{dN_j}{dt}. \tag{11.29}
\]

According to Eq. (11.26), these equations reflect the adaptation of the “permeabilities” \(\gamma_j(t)\). \(\ddot{\nu}_j\) and \(\ddot{\mu}_j\) denote suitable positive parameters.

Apart from adjusting the production rate, it is also possible to adjust the price level \(P_j(t)\) of commodities in sector \(j\). As the adjustment goals are the same, we will assume similar equations:

\[
\frac{1}{P_j(t)} \frac{dP_j}{dt} = \nu_j \left( \frac{N_j^0}{N_j(t)} - 1 \right) - \mu_j \frac{dN_j}{dt}. \tag{11.30}
\]

The use of relative changes guarantees the required non-negativity of prices \(P_j(t) \geq 0\). \(\nu_j\) is an adaptation rate describing the sensitivity to relative deviations of the actual inventory \(N_j(t)\) from the desired one \(N_j^0\), and \(\mu_j\) is a dimensionless parameter reflecting the responsiveness to relative deviations \((dN_j/dt)/N_j(t)\) from the stationary equilibrium state.
Increased prices $P_j(t)$ of products of sector $j$ have a negative impact on the consumption rate $Y_j(t)$ and vice versa. We will describe this by a standard demand function $L_j$ with a negative derivative $L'_j(P_j) = dL_j(P_j)/dP_j$:

$$Y_j(t) = [Y_j^0 + \xi(t)]L_j(P_j(t)). \quad (11.31)$$

This formula takes into account random fluctuations $\xi_j(t)$ over time around a certain average consumption rate $Y_j^0$ and assumes that the average value of $L_j(P_j(t))$ is normalized to one. The fluctuation term $\xi_j(t)$ is introduced here in order to indicate that the variation of the consumption rate is a potentially relevant source of fluctuations.

Inserting (11.31) into (11.32) results in

$$\frac{dN_j(t)}{dt} = Q_j(t) - \sum_{k=1}^m c_{jk}Q_k(t) - \underbrace{[Y_j^0 + \xi(t)]L_j(P_j(t))}_{=Y_j(t)}. \quad (11.32)$$

Further on, we will apply the common linear demand function

$$L_j(P_j) = \max(0, L_j^0 - L_j^1 P_j), \quad (11.33)$$

where $L_j^0$ and $L_j^1$ are non-negative parameters.

**Fig. 11.2.** Illustration of linear and non-linear functions. While linear functions have one maximum in a limited area (left), non-linear functions may have many (local) maxima (right).

### 11.3 Further Reading
Network Effects

12.1 Modelling Network Flows at Nodes

12.1.1 Two Views on Traffic Jams

Let us study the traffic dynamics on the road sections in more detail. Traffic jams can be handled in two different ways: First by determining the number of cars that are delayed compared to free traffic or, second, by determining fronts and ends of traffic jams. The former method is more simple, but it cannot deal correctly with spill-over effects, when the end of a traffic jam reaches the end of a road section. Therefore, the first method is sufficient only in situations where the spatial capacity of road sections is never exceeded.

Method 1: Number of Delayed Vehicles

The first method just determines the difference between the number $N_{i}^{\text{in}}(t)$ of vehicles that would reach the end of road section $i$ upto time $t$ and the number $N_{i}^{\text{out}}(t)$ of vehicles that actually leave the road section upto this time. $N_{i}^{\text{in}}(t)$ just corresponds to the number of vehicles which have entered the road section upto time $t - L_{i}/V_{i}^{0}$, as $L_{i}/V_{i}^{0}$ is the free travel time. This implies

$$N_{i}^{\text{in}}(t) = \int_{0}^{t} dt' A_{i}(t - L_{i}/V_{i}^{0}), \quad (12.1)$$

while the number of vehicles that have actually left the road section upto time $t$ is

$$N_{i}^{\text{out}}(t) = \int_{0}^{t} dt' O_{i}(t). \quad (12.2)$$

Hence, the number $\Delta N_{i}(t)$ of delayed vehicles is given by
\[ \Delta N_i(t) = \int_0^t dt' \left[ A_i(t - L_i/V_i^0) - O_i(t) \right] \geq 0. \quad (12.3) \]

Alternatively, one can use the following differential equation for the temporal change in the number of delayed vehicles:

{delayed} \[ \frac{d \Delta N_i}{dt} = A_i(t - L_i/V_i^0) - O_i(t). \quad (12.4) \]

In contrast, the number of all vehicles on road section \( i \) (independently of whether they are delayed or not) changes in time according to

{all} \[ \frac{d N_i}{dt} = A_i(t) - O_i(t). \quad (12.5) \]

**Method 2: Jam Formation and Resolution**

In our simple macroscopic traffic model, the formation and resolution of traffic jams is described by the shock wave equations, where we have the two characteristic speeds \( V_i^0 \) (the free speed) and \( c \) (the jam resolution speed). According to the theory of shock waves \([14, 17]\), the upstream end of a traffic jam, which is located at a place \( l_i(t) \geq 0 \) upstream of the end of road section \( i \), is moving at the speed

{shock} \[ \frac{dl_i}{dt} = -\frac{A_i(t - [L_i - l_i(t)]/V_i^0) - O_i(t - l_i(t)/c)}{\rho_1(t) - \rho_2(t)} \quad (12.6) \]

with the (free) density

{shock} \[ \rho_1(t) = A_i(t - [L_i - l_i(t)]/V_i^0)/V_i^0 \quad (12.7) \]

immediately before the upstream shock front and the (congested) density

{shock} \[ \rho_2(t) = [1 - TO_i(t - l_i(t)/c)]\rho_{\text{max}} \quad (12.8) \]

immediately downstream of it. This is, because free traffic is upstream of the shock front, and congested traffic downstream of it (for details see Eqs. (1.6) and (1.4) in Ref. [?]). In contrast, the downstream front of a traffic jam is moving at the speed

{shock} \[ \frac{0 - O_i(t - l_i(t)/c)}{\rho_{\text{max}} - O_i(t - l_i(t)/c)/V_i^0} = \frac{O_i(t - l_i(t)/c)}{\rho_{\text{max}} - O_i(t - l_i(t)/c)/V_i^0}, \quad (12.9) \]

since congested traffic with zero flow is upstream of the shock front and free traffic flow occurs downstream of it.
Comparison of the Two Methods

Let us discuss a simple example to make the differences of both descriptions clearer. For this, we assume that, at time $t = 0$, traffic flow on the overall road section $i$ is free, i.e. any traffic jam has resolved and there are no delayed vehicles. The flow shall be stopped by a red traffic light for a time period $t_0$. At time $t = t_0$, the traffic light shall turn green, and the formed traffic jam shall resolve. For the arrival flow, we simply assume a constant value $A_i$, and the road section shall be long enough to take up the forming traffic jam. Moreover, the departure flow shall be $O_i$. Then, according to method 1, the number of delayed vehicles at time $t_0$ is

$$\Delta N_i(t_0) = A_i t_0,$$

and it is reduced according to

$$\Delta N_i(t) = A_i t_0 - (O_i - A_i)(t - t_0).$$

Therefore, any delays are resolved after a time period

$$t - t_0 = \frac{A_i t_0}{O_i - A_i} = \frac{\Delta N_i(t_0)}{O_i - A_i},$$

i.e. at time

$$t_2 = t_0 \frac{O_i}{O_i - A_i}.$$

Afterwards, $\Delta N_i(t) = 0$.

In contrast, the end of the traffic jam grows with the speed

$$\frac{dl_i}{dt} = - \frac{A_i - 0}{A_i/V^0_i - (1 - 0)\rho_{\text{max}}} = \frac{1}{\rho_{\text{max}}/A_i - 1/V^0_i} =: C_i.$$

IST DIESE BEZEICHNUNG ($C_i$) MIT DEN FRÜHEREN KOMPATIBEL?

Therefore, we have $l_i(t_0) = C_i t_0$. Surprisingly, this is greater than $\Delta N_i(t_0)/\rho_{\text{max}}$, i.e. the expected length of the traffic jam based on the number of delayed vehicles. The reason is that the delay of a vehicle joining the traffic jam at location $x_i = L_i - l_i$ is noticed at the downstream end of the road section only after a time period $l_i/V^0_i$.

The resolution of the traffic jam starts from the downstream end with the speed

$$\frac{0 - \hat{Q}_i}{\rho_{\text{max}} - \hat{Q}_i/V^0_i} = \frac{-1}{\rho_{\text{max}}/\hat{Q}_i - 1/V^0_i} = -c,$$

if the outflow is free (i.e. $O_i = \hat{Q}_i$), otherwise with the speed

$$\frac{0 - O_i}{\rho_{\text{max}} - (\rho_{\text{max}} - O_i/c)} = -c,$$
since congested traffic with zero flow and maximum density is upstream of the shock front.

Obviously, the jam resolution has reached the further growing, upstream jam front when \( C_i t = c(t - t_0) \). Therefore, the jam of density \( \rho_{\text{max}} \) has disappeared after a time period \( t - t_0 = C_i t_0 / (c - C_i) \), i.e. at time

\[
t_1 = ct_0 / (c - C_i) .
\]

(12.17)

Surprisingly, it can be shown that \( t_1 < t_2 \), i.e. the traffic jam resolves before the number of delayed vehicles reaches a value of zero. In fact, it still takes the time \( C_i t_1 / \hat{V}_0^i \) until the last delayed vehicle has left the road section, where

\[
\hat{V}_0^i = A_i - O_i
\]

is the shock front between free upstream traffic flow and the congested outflow \( O_i \), which usually differs from the speed \( V_i = O_i / [(1 - TQ_i) \rho_{\text{max}}] \) of outflowing vehicles. For \( O_i = \bar{Q}_i \), we have \( \hat{V}_0^i = V_i^0 \) because of \( 1/c = \rho_{\text{max}} / \bar{Q}_i - 1/V_i^0 \).

Undelayed traffic starts when this shock front reaches the end of the road section, i.e. at time

\[
\hat{V}_0^i = \frac{A_i - O_i}{A_i/V_i^0 - (\rho_{\text{max}} - O_i/c)}
\]

(12.18)

(12.18)

Inserting \( C_i(A_i/V_i^0 - \rho_{\text{max}}) = -A_i \) eventually gives \( t_2 = t_0 O_i/O_i - A_i \). This agrees perfectly with the above result for the first method (based on vehicle delays rather than traffic jams).

In conclusion, both methods of dealing with traffic jams are consistent, and delayed vehicles occur as soon as traffic jam formation begins. However, according to method 1, a queued vehicle at position \( x_i = L_i - l_i \) is counted as delayed only after an extra time period \( l_i/V_i^0 \), but it is counted as undelayed after the same extra time period. This is because method 1 counts on the basis of vehicle arrivals at the downstream end of road section \( i \).

As it is much simpler to use the method 1 based on determining the number of delayed vehicles than using method 2 based on determining the movement of shock fronts, we will use method 1 in the following. More specifically, in Eq. (11.11) we will replace \( l_j(t) < L_j \) by \( \Delta N_j(t) < N_j^{\text{max}} := L_j \rho_{\text{max}} \) and \( l_j(t) = L_j \) by \( \Delta N_j(t) = N_j^{\text{max}} \). This corresponds to a situation in which the vehicles would not queue up along the road section, but at the downstream end of the road section, like in a wide parking lot or on top of each other. As long as road section \( j \) is not fully congested, this difference does not matter significantly. If it is fully congested, the dynamics will potentially be different, defining a modified model of material network flows. However both, the original and the modified model fulfill the conservation equation and show spillover effects.
12.2 Treatment of Merging, Diverging and Intersection Points

While the last section has given general formulas that must be fulfilled at nodes connecting two or more different links, in the following we will give some concrete examples, how to deal with standard elements of street networks. For the treatment of traffic flows at intersections also see Refs. [? , ? , ? , ?].

![Diagram](image)

**Fig. 12.1.** Schematic illustration of the (a) diverging, (b) merging, and (c) intersecting flows discussed in this paper

### 12.2.1 Diverging Flows: One Inflow and Several Outflows

In the case of one road section $i$ diverging into several road sections $j$ (see Fig. 12.1a), Eqs. (??) and (11.13) to (11.15) imply

$$\frac{O_i(t)}{\gamma_i(t)} \leq \min \left\{ \gamma_i(t) \max \left[ \hat{Q}_i \Theta(\Delta N_i > 0), A_i \left( t - \frac{L_i}{V_i} \right) \right], \min_j \left[ \frac{I_j'}{I_i \alpha_{ij}} \max \left( \hat{Q}_j \Theta(l_j < L_j), O_j(t - L_j/c) \right) \right] \right\} \quad (12.20)$$

for all $i$. If we assume that downstream road sections are never completely congested, this simplifies to

$$O_i(t) = \min \left\{ Q_i, \gamma_i \max \left[ \hat{Q}_i \Theta(\Delta N_i > 0), A_i \left( t - \frac{L_i}{V_i} \right) \right] \right\} \quad (12.21)$$

with

$$Q_i = \min_j \left( \frac{I_j' \hat{Q}_j}{I_i \alpha_{ij}} \right) \quad (12.22)$$

Otherwise

$$Q_i(t) = \min_j \left[ \max \left( \frac{I_j' \hat{Q}_j}{I_i \alpha_{ij}} \Theta(l_j < L_j), \frac{I_j' O_j(t - L_j/c)}{I_i \alpha_{ij}} \right) \right] \quad (12.23) \quad \{ \text{otherwise} \}$$
12.2.2 Merging Flows: Two Inflows and One Outflow

We assume a flow $I_1 O_1(t)$ that splits into two flows $I_1 O_1(t)\alpha_{11}$ (going straight) and $I_1 O_1(t)\alpha_{12}$ (turning right), but a right-turning flow $I_2 O_2(t)$ merging with flow $I_1 O_1(t)\alpha_{11}$, as in turn-right-on-red setups (see Fig. 12.1b). For this situation, we have the equations

$$I_1' A_1'(t) = I_1 O_1(t)\alpha_{11} + I_2 O_2(t) \leq I_1' \hat{A}_1'(t),$$  \hspace{1cm} (12.24)

$$I_2' A_2'(t) = I_1 O_1(t)\alpha_{12} \leq I_2' \hat{A}_2'(t).$$  \hspace{1cm} (12.25)

One can derive

$$0 \leq O_1 = \min \left[ \hat{O}_1(t), \frac{I_1' \hat{A}_1'(t) - I_2 O_2(t)}{I_1\alpha_{11}}, \frac{I_2' \hat{A}_2'(t)}{I_1\alpha_{12}} \right]$$  \hspace{1cm} (12.26)

and

$$0 \leq O_2 = \min \left[ \hat{O}_2(t), \frac{I_1' \hat{A}_1'(t) - I_1 O_1(t)\alpha_{11}}{I_2} \right].$$  \hspace{1cm} (12.27)

Let us set

{speci}  
$$O_1 = \min \left[ \hat{O}_1(t), \frac{I_1' \hat{A}_1'(t)}{I_1\alpha_{11}}, \frac{I_2' \hat{A}_2'(t)}{I_1\alpha_{12}} \right]$$  \hspace{1cm} (12.28)

and

{maxi}  
$$O_2(O_1) = \min \left[ \hat{O}_2(t), \frac{I_1' \hat{A}_1'(t) - I_1 O_1\alpha_{11}}{I_2} \right].$$  \hspace{1cm} (12.29)

Then, it can be shown that $O_2(t) \geq 0$ and $O_1(t) \leq [I_1' \hat{A}_1'(t) - I_2 O_2(t)]/(I_1\alpha_{11})$, as demanded. If $O_1(t)$ is chosen a value $\Delta O_1$ smaller than specified in Eq. (12.28), but $O_2$ is still set to the maximum related value $O_2(O_1 - \Delta O_1)$ according to Eq. (12.29), the overall flow

$$F = I_1 O_1 + I_2 O_2$$  \hspace{1cm} (12.30)

is reduced as long as $\alpha_{11} < 1$, since this goes along with additional turning flows (while the number of lanes does not matter!). Therefore, it is optimal to give priority to the outflow $O_1(t)$ according to Eq. (12.28) and to add as much outflow $O_2(t)$ as capacity allows. This requires suitable flow control measures, otherwise the optimum value of the overall flow $F$ could not be reached. In fact, the merging flow would “steel” some of the capacity reserved for the “main” flow ($i = 1$), which would reduce the possible outflow $O_1(t)$ and potentially cause a breakdown of free traffic flow, as it is known from on-ramp areas of freeways [7].

12.2.3 A Side Road Merging with a Main Road

Compared to the last section, the situation simplifies, if we have just a side road or secondary turning flow merging with a the flow of a main road without
any turning flow away from the main road. In this case, we have \( \alpha_{11} = 1 \) and \( \alpha_{12} = 0 \), which leaves us with the relationships

\[
O_1 = \min \left[ \hat{O}_1, \frac{I_1 \hat{A}_1(t)}{I_1} \right] \tag{12.31}
\]

and

\[
O_2(O_1) = \min \left[ \hat{O}_2(t), \frac{I_1 \hat{A}_1(t) - I_1 O_1}{I_2} \right]. \tag{12.32}
\]

according to Eqs. (12.28) and (12.29).

### 12.3 Effects of Several On- and Off-Ramps along Freeways

We are now prepared to come back to the treatment of freeway traffic flows. Freeways are not at all spatially homogeneous. Due to the existence of on- and off-ramps, gradients, curves etc., a freeway is composed of road sections of different capacities, even if the number of lanes is constant. If the capacity is reduced from one road section to the next one in downstream direction, we speak of a bottleneck. Upstream of bottlenecks, there is a danger of queue formation or congestion, if the traffic volume becomes too high, while downstream of the upstream end of a bottleneck, one mostly observes free traffic, if there is not another activated bottleneck further downstream.

The phase diagrams depicted in Fig. 7.2 assume a single bottleneck only. In principle, however, phase diagrams could also be generated for more complex freeway geometries with several bottlenecks. These would be multidimensional (with one additional dimension per bottleneck). For this reason, there are many more possibilities to classify different congested traffic states for freeways with several bottlenecks.

Let us call a bottleneck “activated” if a vehicle queue, i.e. congestion has formed upstream. The vehicle queue grows if the inflow into the congested area is larger than the outflow (as typically during rush hours), otherwise it shrinks (see .......). In the first case, the congested area upstream of some bottleneck \( i + 1 \) may eventually reach a freeway section \( i \) with a higher flow capacity (e.g., if there is an off-ramp or a larger number of lanes) or a lower one (e.g., if there is a smaller number of lanes, an on-ramp, or a gradient). In the moment when it reaches this section, it activates a bottleneck around the downstream end of the upstream road section with higher or lower flow capacity (until the congestion passes the downstream end in driving direction).

For a quantitative description, we can use the formalism of Sec. ?? in order to simplify our discussion, let us assume a freeway for which the bottlenecks are constituted by ramp flows \( \Delta Q_j \) only, where positive flows correspond to on-ramps and negative ones to off-ramps. Moreover, let us assume that the number of freeway lanes is fixed and that a ramp either enters or exits at
the end of each freeway section \( j \). Moreover, the ramps shall not provide additional lanes to the freeway over a significant length (as they usually do). The flow capacity \( Q_j^{\text{act}}(t) \) of the bottleneck at the downstream end of section \( j \), if activated \((l_j(t) > 0)\), is then limited by \( \hat{Q}_j \), which depends on the desired velocity \( V_j^0 \) according to Eq. (12.33). Moreover, considering flow conservation,

\[
A_{j+1}(t) = O_j(t) + \Delta Q_j(t) ,
\]

(12.33)

the flow capacity \( Q_j^{\text{act}}(t) \) is also limited by the potential inflow \( \hat{A}_{j+1}(t) \) into the subsequent freeway section, minus the flow capacity which is used by the ramp flow \( \Delta Q_j(t) \). Altogether this implies

\[
Q_j^{\text{act}}(t) = \min \left[ \hat{A}_{j+1}(t) - \Delta Q_j(t), \hat{Q}_j \right]
\]

(12.34)

\[
= \begin{cases} 
\min \left[ \hat{Q}_{j+1} - \Delta Q_j(t), \hat{Q}_j \right] & \text{if } l_{j+1}(t) < L_{j+1}, \\
\min \left[ O_{j+1}(t) - L_{j+1}/c - \Delta Q_j(t), \hat{Q}_j \right] & \text{otherwise}.
\end{cases}
\]

Note that, in case of an off-ramp flow \( \Delta Q_j(t) < 0 \), capacity is added, but only if the downstream freeway section \( j + 1 \) is fully congested \((i.e., l_{j+1}(t) = L_j)\).

From the above formula, we can determine the effective bottleneck strength \( \Delta Q_j^{\text{eff}}(t) = \hat{Q}_j - Q_j^{\text{act}}(t) \) as

\[
\Delta Q_j^{\text{eff}}(t) = \max \left[ \hat{Q}_j - \hat{A}_{j+1}(t) + \Delta Q_j(t), 0 \right]
\]

(12.35)

\[
= \begin{cases} 
\max \left[ \hat{Q}_j - \hat{Q}_{j+1} + \Delta Q_j(t), 0 \right] & \text{if } l_{j+1}(t) < L_{j+1}, \\
\max \left[ \hat{Q}_j - O_{j+1}(t) - L_{j+1}/c + \Delta Q_j(t), 0 \right] & \text{otherwise},
\end{cases}
\]

which can never be negative. If \( \hat{Q}_{j+1} = \hat{Q}_j = \hat{Q} \) (i.e. the desired velocities \( V_j^0 \) are the same), this formula simplifies and becomes

\[
\Delta Q_j^{\text{eff}}(t) = \begin{cases} 
\max \left[ \Delta Q_j(t), 0 \right] & \text{if } l_{j+1}(t) < L_{j+1}, \\
\max \left[ \hat{Q} - O_{j+1}(t) - L_{j+1}/c + \Delta Q_j(t), 0 \right] & \text{otherwise}.
\end{cases}
\]

(12.36)

That is, if the downstream freeway section is not fully congested, the bottleneck strength is given by the on-ramp flow \( \Delta Q > 0 \) (or it is zero in case of an off-ramp). However, if congestion on the downstream freeway section \( j + 1 \) reaches the downstream end of section \( j \), the bottleneck strength is given by \( \max[\hat{Q}_j - O_{j+1}(t) - L_{j+1}/c + \Delta Q_j(t), 0] \), i.e. by the difference between the maximum outflow capacity \( \hat{Q}_j \) and the outflow of the downstream freeway section at time \( t - L_{j+1}/c \) (as variations in the outflow of section \( j + 1 \) propagate upstream at speed \( c \)). Additionally, the ramp-flow \( \Delta Q_j(t) \) contributes to the bottleneck strength. It is increased by an on-ramp (which consumes some of the freeway capacity), while the bottleneck strength is decreased by an off-ramp. That is congestion upstream of an off-ramp should be either less serious (if the downstream freeway section is fully congested) or non-existent (if it is not).
In conclusion, we expect PLC, MLC, SGW, OCT, or HCT states separated by free traffic, if none of the freeway sections is fully congested. Otherwise, we expect combinations of several congestion states in the course of the freeway. These combinations may have new appearances. In the following, we will discuss two examples.

### 12.3.1 Combination of an On-Ramp with an Off-Ramp further Upstream

For illustration, let us first focus on the combination of an off- with an on-ramp, which is typical for the freeway design in Germany and other countries (see Fig. 12.2 for a sketch). For simplicity, we will assume $\hat{Q}_{j+1} = \hat{Q}_j = \hat{Q}$.

Obviously, congestion will start to occur at the stronger bottleneck, i.e. at the location of the on-ramp. The flow downstream of the activated on-ramp bottleneck $j+1$ is $A_{j+2}(t) = \hat{Q}$, if road section $j+2$ is not fully congested. The freeway flow immediately upstream of the on-ramp is

$$O_{j+1}(t) = A_{j+2}(t) = \hat{Q} - \Delta Q_{j+1}(t) \leq \hat{Q}. \quad (12.37)$$

To avoid confusion, for the off-ramp it is important to remember $+\Delta Q_j(t) = -|\Delta Q_j(t)| \geq 0$. We will have growing congestion in freeway section $j$, if the arrival flow $A_j = O_{j+1}(t)$ exceeds the departure flow $O_j(t)$ (approximately). As the departure flow $O_j(t)$ is given by the previous arrival flow $A_j(t - L_j/V_0)$ minus the off-ramp flow $|\Delta Q_j(t)|$, the condition for spatially extending congestion reads

$$A_{j+1}(t) = O_j(t) - |\Delta Q_j(t)| = A_j(t - L_j/V_j^0) - |\Delta Q_j(t)| > O_{j+1}(t) . \quad (12.38)$$

When the upstream congestion front finally reaches the off-ramp, the arrival flow $A_{j+1}(t)$ is limited by the downstream flow:

$$A_{j+1}(t) = O_j(t) - |\Delta Q_j(t)| = O_{j+1}(t - L_{j+1}/c) \leq O_j(t) . \quad (12.39)$$

If $A_j(t) > O_j(t)$ (approximately), congestion will even grow upstream of the off-ramp.

What is the overall traffic pattern that we expect to see in this scenario? Obviously, we will have a spatially extended form of congested traffic between the on-ramp and the upstream off-ramp. Inserting Eq. (12.37) into Eq. (12.36), the bottleneck strength at the location of the off-ramp turns out to be

$$\Delta Q_{j+1}^{\text{eff}}(t) = \begin{cases} 0 & \text{if } l_{j+1}(t) < L_{j+1}, \\ \max \left[ \Delta Q_{j+1}(t - L_{j+1}/c) - |\Delta Q_j(t)|, 0 \right] & \text{otherwise}. \quad (12.40) \end{cases}$$

At least on average, this is obviously smaller than (or equal to) the bottleneck strength at the location of the on-ramp which is

$$\Delta Q_{j+1}^{\text{eff}}(t) = \Delta Q_{j+1}(t) \quad (12.41)$$
a) \[ Q_{up} - \Delta Q_{off} = Q_{up} > Q_{cong} = Q_{out} - \Delta Q_{on} \]

b) \[ Q_{cong} = Q_{out} + \Delta Q_{off} > Q_{out} = Q_{cong} = Q_{out} - \Delta Q_{on} \]

Fig. 12.2. BEZEICHNUNGEN ANPASSEN! Illustration of our notations for a combination of an on-ramp bottleneck with an upstream off-ramp. Traffic flow is from left to right. (a) When the upstream flow \( Q_{up} \) upstream the off-ramp minus the off-ramp flow \( \Delta Q_{off} \) exceeds the outflow \( Q_{out} \) from congested traffic minus the on-ramp flow \( \Delta Q_{on} \), congested traffic upstream of the on-ramp (grey area) is expected to grow. (b) As soon as the congested area extends up to the location of the off-ramp, the off-ramp bottleneck is activated. Its outflow \( Q^{'\text{off}}_{out} \) is given by the congested flow \( Q_{cong} \) upstream of the on-ramp, while the congested flow \( Q^{'\text{cong}}_{cong} \) upstream of the off-ramp is higher by the amount \( \Delta Q_{off} \) corresponding to the off-ramp flow. The flow upstream of the congestion and the off-ramp is denoted by \( Q^{'\text{up}}_{up} \).

\[ \{2\text{ramps}\} \]

according to Eq. (12.36), as long as freeway section \( j + 2 \) is not fully congested.

In conclusion, we expect a milder form of congestion upstream of the off-ramp, which implies that the overall congestion pattern at combinations of on-and off-ramps may have a new appearance (as compared to the elementary forms of congestion discussed in Sec. ...). For example, one may find homogeneous or oscillating congested traffic between the on-ramp and the off-ramp, but stop-and-go waves upstream of the on-ramp. Such a pattern is, in fact, frequently observed.\(^1\)

Finally, note that the above treatment of effective off-ramp bottlenecks does not take into account bottleneck effects upstream of off-ramps, which are sometimes caused by weaving flows, i.e. lane-changes by drivers who want to leave the freeway. Such effects can be easily taken into account by specifying freeway capacities or bottleneck strengths as a function of the upstream freeway flow and the off-ramp flow.

\(^1\) In the terminology of Kerner, it is called a “general pattern” [?].
12.3 Effects of Several On- and Off-Ramps along Freeways

12.3.2 Combination of a Strong Bottleneck with a Weak One Downstream

The following scenario will assume a situation where we have two successive bottlenecks, a weaker one downstream and a stronger one upstream. While the strong bottleneck could be a freeway intersection, the weak one could be a gradient along the freeway, which we may attribute some effective bottleneck strength as well. It could also be a speed limit, as formula (??) implies that the outflow $\hat{Q}_{j+1}(t)$ from congested traffic decreases with a reduction in the desired speed $V_j^0$. For simplicity, however, we will assume $\hat{Q}_{j+1} = \hat{Q}_j = \hat{Q}$, again.

Furthermore, let the strength of the downstream bottleneck be
\[
\Delta Q_{j+1}(t) < Q_{\text{max}} - \hat{Q} \ll \hat{Q}.
\] (12.42)

We will now assume a rush-hour situation in which the outflow from section $O_j(t)$ has exceeded the outflow $\hat{Q}$ from congested traffic, but together with the on-ramp flow $\Delta Q_j(t)$, it shall still be below the maximum flow $Q_{\text{max}}$, i.e.
\[
\hat{Q} + \Delta Q_j(t) < Q_{\text{max}}.
\] (12.43)

Therefore, we have free flow as long as $\Delta Q_j(t)$ is small enough. However, as the on-ramp flow $\Delta Q_j(t)$ grows large, the flow
\[
A_{j+2}(t) = O_{j+1}(t) + \Delta Q_{j+1}(t) = [O_j(t) + \Delta Q_j(t)] + \Delta Q_{j+1}(t)
\] (12.44)
will eventually exceed the maximum flow $Q_{\text{max}}$ and the free flow will break down. Surprisingly, this happens first behind the weak bottleneck, since it is passed by the highest flow (the main flow $A_j(t)$ plus both on-ramp flows).

After the breakdown of free flow, the flow $A_{j+1}(t)$ downstream of the second bottleneck will be $\hat{Q}$, while the upstream flow immediately upstream will be
\[
O_{j+1}(t) = \hat{Q} - \Delta Q_{j+1}(t).
\] (12.45)

As the inflow $A_{j+1}(t)$ into the section between both bottlenecks is now greater than the considerably reduced outflow $O_{j+1}(t)$, we will have quickly extending congestion, which eventually reaches the upstream bottleneck. This will cause a breakdown of the traffic flow at the stronger bottleneck. From this moment on, the traffic flow arriving from the upstream bottleneck is
\[
A_{j+1}(t) = O_{j+1}(t - L_{j+1}/c) = \hat{Q} - \Delta Q_{j+1}(t - L_{j+1}/c),
\] (12.46)
and the departure flow upstream of it is
\[
O_j(t) = A_{j+1}(t) - \Delta Q_j = \hat{Q} - \Delta Q_{j+1}(t - L_{j+1}/c) - \Delta Q_j(t),
\] (12.47)
i.e. the flow upstream of the first bottleneck is smaller and the density higher.
In the following, we will focus on the situation upstream of the second bottleneck. As we have assumed a small bottleneck strength $\Delta Q_{j+1} \ll \hat{Q}$, the upstream flow is
\[ O_{j+1}(t) = \hat{Q} - \Delta Q_{j+1}(t) \approx \hat{Q}. \] (12.48)

Let $\rho_{\text{cong}}$ be the density on the congested part of the fundamental diagram with
\[ Q_e(\rho_{\text{cong}}) = \hat{Q}. \] (12.49)

Then, $V_e(\rho_{\text{cong}})$ is the speed at which vehicles move between both bottlenecks. Note that, although the flow $O_{j+1}(t)$ almost has values of the (free) outflow from congested traffic, the associated speed $V_e(\rho_{\text{cong}})$ is considerably slower: it belongs to the congested branch of the fundamental diagram, while the outflow $\hat{Q}$ is related with a speed on the free flow branch.\(^2\) Moreover, even if the density $\rho_{\text{cong}}$ belongs to the unstable traffic regime, density variations will be small close to the weak bottleneck, given that its bottleneck strength $\Delta Q_{j+1}(t)$ is constant or very small. They could only increase in amplitude while travelling upstream at the speed $c$ of perturbations in congested traffic.

In Sec. ??, we will see that this finding is relevant to understand the “homogeneous-in-speed states” discovered by Kerner [?]. Here, we conclude this section by noting that no bottleneck is so weak that it can be neglected. \textit{Even the smallest bottleneck strength can cause a breakdown of traffic flow, if combined with another bottleneck, and it can reduce the vehicle speed considerably, even if it is associated almost with free flow upstream.}

12.3.3 Spill-Over Effects

\(^2\) The speed corresponding to the outflow from congested traffic amounts to 85 km/h on German freeways, while $V_e(\rho_{\text{cong}}) \approx 60$ km/h.
Basics of Intersection Control

Until now, we have assumed that the time-dependent permeabilities $\gamma_j(t)$ are given. One may, however, also think of manipulating $\gamma_j(t)$ in order to optimize some goal function. In fact, when the traffic volume is high enough, it can have positive effects to operate traffic flows at intersections in an oscillatory way, as it is usually done by means of traffic lights. The red and amber light (corresponding to $\gamma_j(t) = 0$) cause vehicles to queue up, but this implies a high flow rate and an efficient service of vehicles when the traffic light turns green (i.e. $\gamma_j(t) = 1$). By the related organization of the traffic flow into vehicle platoons, ideally it becomes possible to coordinate vehicle arrivals in a way that platoons from different directions can pass the intersection at different times without interfering with other vehicle streams. The great challenge, therefore, is to coordinate vehicle platoons so that the goal function is optimized.

13.1 Specification of Goal Functions

13.1.1 Overall Throughput

One natural concept of traffic flow optimization is to maximize the average overall throughput of the system. This would be measured by the goal function

$$G_t(t) = \frac{1}{t} \sum_j \int_0^t dt' \ O_j(t').$$

(13.1)

According to Eq. (13.1) it is obvious, that $G_t(t)$ depends on the inflow into the system (and the outflow as well). This makes the measure basically dependent on the time-dependent origin-destination matrices of vehicle flows (see Sec. .......). For a given origin-destination matrix, there will normally be many possible traffic light controls with the same throughput, which is typical for a combinatorial optimization problem. We will see the reasons for this in Sec. ??.
13.1.2 Homogeneously Weighted Waiting Times

A more sensitive differentiation between different possible signal control schemes is made by the cumulative travel time or, alternatively, the cumulative waiting time

\[
G_w(t) = \frac{1}{t} \sum_j \int_0^t dt' \frac{dW_j}{dt} = \frac{1}{t} \sum_j I_j \int_0^t dt' \Delta N_j(t').
\] (13.2)

This expression quantifies the time average of the overall delay time. The term on the right-hand side describes the increase of the overall waiting time proportionally to the number \(\Delta N_j\) of delayed cars.

For later considerations, we also introduce the generalized

\[
G_a(t) = \frac{1}{t} \sum_j I_j \int_0^t dt' \left( \Delta N_j(t') + T_j' C_j \rho_{\text{max}} \Theta(\Delta N_j > 0) \right),
\] (13.3)

where \(C_j \rho_{\text{max}} \Theta(\Delta N_j > 0)\) is the number of stopped vehicles per unit time according to Eq. (??). Therefore, the second term considers additional delays due to stopping vehicles, and it corrects the macroscopic network flow model proposed in Sec. ?? regarding its assumption of immediate vehicle acceleration. In reality, a finite vehicle acceleration \(a\) causes additional delays of \(V_j^0/(2a)\), and the reaction time \(T_r\) must be considered as well. This leads to

\[
T_j' = T_r + \frac{V_j^0}{2a},
\] (13.4)

where \(T_r\) is assumed to be about the same as the safe time gap \(T\). (Note that delays \(V_j^0/(2b)\) due to a finite deceleration \(b\) do not additionally contribute to the delay times, as it does not matter whether delayed vehicles spend their time decelerating or stopped. The finite deceleration only matters slightly, when the exact moment must be determined when a road section becomes completely congested.)

13.1.3 Exponentially Weighted Waiting Times

As the prediction of the future traffic dynamics becomes less reliable with the forecast time horizon \(1/\lambda\) and the optimization cannot be performed over an infinite time, it makes sense to weight future waiting times exponentially with a factor \(\lambda e^{-\lambda t}\). Instead of (13.3), one would then minimize the function

\[
G_e = \sum_j \int_0^\infty dt \lambda e^{-\lambda t} \frac{dW_j}{dt} = \sum_j \lambda I_j \int_0^\infty dt \ e^{-\lambda t} \left[ \Delta N_j(t) + T_j' C_j \rho_{\text{max}} \Theta(\Delta N_j > 0) \right].
\] (13.5)
While this approach is less suitable for an analytical optimization, it reminds of formulations of discounted functions in economics and is perfectly suited for a computer-based evaluation. Note however that, assuming piecewise constant flows, the function $G'$ can be further evaluated............ (See appendix and pages ε′ ff).

### 13.2 Optimization of One Traffic Light with Constant Inflows

For matters of simplicity, let us focus here on a traffic flow control with two signal phases only. This is in principle sufficient to reach all points in the road network, as the traffic organization in Barcelona shows: Just assume unidirectional flows in all streets with alternating directions. Then, in each phase, traffic flows straight and/or turns (either right or left, depending on the respective intersection). Hence, two intersecting unidirectional roads imply two possible traffic phases, which alternate. That is, the sequence of phases is completely determined, while the duration of the phases may vary. For constant inflows, one expects a periodic optimal traffic light schedule, at least after a transient time period.

#### 13.2.1 Periodic Operation

Let us assume constant inflows $A_j$ for simplicity. Moreover, let $\Delta T_j$ denote the green time period for road section $j$ and $\tau'$ the amber time period or, in other words, the switching or setup time between the service of different flow directions. Then, the cycle time

$$T_{\text{cyc}} = \Delta T_1 + \Delta T_2 + 2\tau'.$$

(13.6)

The numbers of vehicles accumulating during the red and amber time periods are

$$I_1 \Delta N_{1_{\text{max}}} = I_1 A_1 (\Delta T_2 + 2\tau') \quad \text{and} \quad I_2 \Delta N_{2_{\text{max}}} = I_2 A_2 (\Delta T_1 + 2\tau').$$

(13.7)  \text{from1}

According to Eq. (??), the minimum green times required to serve all vehicles that have arrived until the queue has completely dissolved are given by

$$T_j = \frac{\Delta N_{j_{\text{max}}}}{Q - A_j}.$$  \text{from2}

This formula takes into account that further vehicles are arriving during the service of the vehicle queue by a green light. From Eqs. (13.7) and (13.8) we obtain

$$\Delta T_1 = \frac{A_1}{Q_1 - A_1} \left[ \frac{A_2 (\Delta T_1 + 2\tau')}{Q_2 - A_2} + 2\tau' \right] = 2\tau' \frac{A_1}{Q_1 - A_1} \frac{1 + \frac{A_2}{Q_2 - A_2}}{1 - \frac{A_1 A_2}{(Q_1 - A_1)(Q_2 - A_2)}}.$$  \text{(13.9)
With the analogous formula for $\Delta T_2$ we can determine the related cycle time, if the traffic light turns red immediately when all queued vehicles have been served. After a few intermediate mathematical steps we finally get

$$T_{\text{cyc}} = \frac{2\tau'}{1 - \frac{A_1}{\hat{Q}_1} - \frac{A_2}{\hat{Q}_2}}.$$  \hspace{1cm} (13.10)

We can see that the cycle time diverges in the limit

$$\frac{A_1}{\hat{Q}_1} + \frac{A_2}{\hat{Q}_2} \to 1.$$  \hspace{1cm} (13.11)

If this expression (13.10) becomes negative, the cycle time is not anymore defined. This case corresponds to situations where the vehicle queues in one or both ingoing road sections are growing larger and larger in time, as the intersection does not have enough capacity to serve both arrival flows.

Remember that the average throughput of the intersection is given by the overall flow of vehicles during one cycle time. If all arriving vehicles are served (i.e. in the case where we do not have infinitely growing vehicles queues over time), the throughput is given by

$$G_t = \frac{(I_1 A_1 + I_2 A_2)(\Delta T_1 + \Delta T_2 + 2\tau')}{T_{\text{cyc}}} = I_1 A_1 + I_2 A_2.$$  \hspace{1cm} (13.12)

That is, as long as the intersection capacity is not reached, the throughput is determined by the arrival flows $A_j$, and it is obvious that excess green times

$$\Delta t_j = \Delta T_j - T_j$$  \hspace{1cm} (13.13)

with $0 \leq \Delta t_j < \infty$ do not lead to smaller or larger intersection throughputs.

However, the intersection throughput is limited by the maximum intersection capacity $G^\text{max}_t$, which is given by the maximum number of vehicles that can be served during the green time periods $\Delta T_j$, i.e.

$$G^\text{max}_t = \frac{I_1 \hat{Q}_1 \Delta T_1 + I_2 \hat{Q}_2 \Delta T_2}{T_{\text{cyc}}} = \frac{I_1 \hat{Q}_1 \Delta T_1 + I_2 \hat{Q}_2 \Delta T_2}{\Delta T_1 + \Delta T_2 + 2\tau'}.$$  \hspace{1cm} (13.14)

Obviously, larger green times $\Delta T_j$ can slightly increase the intersection capacity during periods of high arrival flows. Therefore, green time periods are usually increased during rush hours. However, this also increases the average waiting times. Moreover, extending green times reduce the intersection capacity, as soon as road sections that vehicles are trying to enter are fully congested.

Note that the maximum intersection capacity may depend significantly on the green time fractions $\Delta T_j/T_{\text{cyc}}$, if the outflow capacities $\hat{Q}_j$ or the number $I_j$ of lanes happen to be different. In this case, we find

$$\lim_{T_{\text{cyc}} \to \infty} G^\text{max}_t \left(\frac{\Delta T_1}{T_{\text{cyc}}}\right) = \begin{cases} I_1 \hat{Q}_1 & \text{for } \frac{\Delta T_1}{T_{\text{cyc}}} \to 1, \\ I_2 \hat{Q}_2 & \text{for } \frac{\Delta T_1}{T_{\text{cyc}}} \to 0. \end{cases}$$  \hspace{1cm} (13.15)
However, if the vehicle flow is not constant, but characterized by platoons, the phase of traffic light control can be much more significant for intersection capacity than the green time fractions or the cycle time. Therefore, the mutual coordination of neighboring traffic lights has a major impact (see Sec. ??).

13.2.2 “Slower-Is-Faster Effect” I

In the following we will show that, from the point of view of flow optimization, it may be reasonable to wait for some additional time \( \Delta t \) until turning green for road section 1, even if the vehicle queue in road section 2 has been totally dissolved at time \( t = 0 \). If we turn green at time \( \Delta t \), there will be no flow for the amber time period \( \tau' \), and afterwards a maximum flow of \( I_1 \hat{Q}_1 \) for a time period

\[
T_1 = \frac{\Delta N_1(0) + A_1(\Delta t + \tau')}{\hat{Q}_1 - A_1}. \tag{13.16}
\]

Switching the traffic light at time \( \Delta t \) will only increase the average flow, if the average flow until time \( t = (\Delta t + \tau' + T_1) \) exceeds the average flow \( I_2 A_2 \) when serving road section 2 all the time. That is, the condition for switching the traffic light from the perspective of the maximization of average flow is

\[
\frac{I_2 A_2 \Delta t + I_1 Q_1 T_1}{\Delta t + \tau + T_1} > I_2 A_2 \tag{13.17}
\]

or

\[
I_1 Q_1 - I_2 A_2 > \frac{I_2 A_2 \tau (\hat{Q}_1 - A_1)}{\Delta N_1(0) + A_1 (\Delta t + \tau')}. \tag{13.18}
\]

That is, no switching is recommended from the perspective of flow optimization, if \( I_1 \hat{Q}_1 \leq I_2 A_2 \), i.e. if the overall maximum outflow \( I_1 \hat{Q}_1 \) from road section 1 is smaller than or equal to the overall inflow \( I_2 A_2 \) into road section 2. Otherwise, it is recommended to turn red immediately after dissolving the vehicle queue in road section 2 (\( \Delta t = 0 \)), if

\[
I_1 Q_1 - I_2 A_2 > \frac{I_2 A_2 \tau (\hat{Q}_1 - A_1)}{\Delta N_1(0) + A_1 \tau'}. \tag{13.19}
\]

If this condition is not satisfied, but \( I_1 Q_1 > I_2 A_2 \), there is always some value \( \Delta t > 0 \), for which condition (13.18) is fulfilled. In such cases, an excess green time \( \Delta t > 0 \) is recommended for road section 2. This flow-optimizing strategy of delayed switching may be interpreted as a “slower-is-faster effect”.

Such strategies are quite common, when flows in networks are optimized (see also Secs. ?? and ??).

WHAT ABOUT THE WAITING TIMES???
13.2.3 The Optimize-One-Phase Approach

We have seen that the throughput is independent of the excess green times $\Delta t_j$, as long as the intersection capacity is not reached. This implies that we can apply an additional optimization criterium. Therefore, let us now focus on the minimization of the cumulative waiting time.

When minimizing the goal function $G_w(t)$, it is essential up to what time $t$ we extend the integral. In principle, it is possible to integrate over a full cycle or even many cycles of traffic operation, but the resulting formulas do not provide an intuitive understanding anymore. We will, therefore, focus on the optimization of a single phase, with a full amber phase of time period $\tau'$ in the beginning and at the end. After completion of a green phase, the same process is repeated, so that one can make up the traffic light schedule as a sequence of optimized single phases. It should be stated, however, that the optimization of multi-phase setups may yield even better results. Moreover, the treatment of situations with varying or even pulsed traffic flows is more difficult than discussed below. This issue will be addressed in Sec. ??.

Let us again denote the duration of the green phase by $\Delta T_1$. Our goal is to minimize the function

$$G_w(\Delta T_1 + 2\tau) = \frac{F_w(\Delta T_1 + 2\tau)}{\Delta T_1 + 2\tau}$$  \hspace{1cm} (13.20)

with

$$F_w(t) = \sum_j I_j \left( \int_0^t dt' \int_0^{t'} [A_j - O_j(t'')] \right)$$  \hspace{1cm} (13.21)

and a step-wise constant outflow $O_j = \hat{Q}_j$ or $O_j = A_j$. We will distinguish two cases: (i) The green time is potentially terminated before all queued vehicles have been served, i.e. $\Delta T_i \leq T_i$. In this case, using the abbreviation

$$\Delta N_{1}^{\text{max}} = \Delta N_1(0) + A_1 \tau$$  \hspace{1cm} (13.22)

we have

$$F_w(\Delta T_1 + 2\tau) = I_1 \left\{ \Delta N_1(0)\tau + A_1 \frac{\tau'^2}{2} + \Delta N_1^{\text{max}} \Delta T_1 - (\hat{Q}_1 - A_1) \frac{\Delta T_1^2}{2} \right. \right.$$

$$+ \left. [\Delta N_1^{\text{max}} - (\hat{Q}_1 - A_1) \Delta T_1] \tau + A_1 \frac{\tau'^2}{2} \right\}$$

$$+ I_2 \left[ \Delta N_2(0)(\Delta T_1 + 2\tau') + A_2 \frac{(\Delta T_1 + 2\tau)^2}{2} \right]$$

$$= I_1 \left\{ \Delta N_1(0)(\Delta T_1 + 2\tau') + \hat{Q}_1 \tau' (\Delta T_1 + 2\tau') - (\hat{Q}_1 - A_1) \frac{(\Delta T + 2\tau)^2}{2} \right\}$$

$$+ I_2 \left[ \Delta N_2(0)(\Delta T_1 + 2\tau') + A_2 \frac{(\Delta T_1 + 2\tau)^2}{2} \right].$$ \hspace{1cm} (13.23)
After division by \((\Delta T_1 + 2\tau')\) we stay with
\[
G_w(\Delta T_1 + 2\tau) = I_1 \left[ \Delta N_1(0) + \bar{Q}_1 \tau - (\bar{Q}_1 - A_1) \frac{\Delta t + 2\tau}{2} \right] \\
+ I_2 \left[ \Delta N_2(0) + A_2 \frac{\Delta T_1 + 2\tau}{2} \right]. \tag{13.24}
\]
If \(I_1(\bar{Q}_1 - A_1) < I_2 A_2\), the minimum of this function is reached for \(\Delta T_1 = 0\), corresponding to a situation where it is not favorable to turn green for section \(j = 1\). This agrees with the above result for flow optimization. For \(I_1(\bar{Q}_1 - A_1) > I_2 A_2\), the value of \(G_w\) goes down with growing values of \(\Delta T_1\), and the minimum is reached at the boundary \(\Delta T_1 = T_1\).

(ii) Let us assume that we (potentially) have an excess green time, i.e. \(\Delta t_i = \Delta T_i - T_i \geq 0\). In this case, with
\[
T_1 = \frac{N_{1\text{max}}}{\bar{Q}_1 - A_1} = \frac{N_1(0) + A_1 \tau}{\bar{Q}_1 - A_1}, \tag{13.25}
\]
we have
\[
F_w(\Delta T_1 + 2\tau) = I_1 \left[ \Delta N_1(0) \tau + A_1 \frac{\tau'^2}{2} + \Delta N_{1\text{max}} T_1 - (\bar{Q}_1 - A_1) \frac{T_1^2}{2} + A_1 \frac{\tau'^2}{2} \right] \\
+ I_2 \left[ \Delta N_2(0)(\Delta T_1 + 2\tau') + A_2 \frac{(\Delta T_1 + 2\tau')^2}{2} \right] \\
= I_1 \left[ N_{1\text{max}} \tau' + \frac{(N_{1\text{max}})^2}{2(\bar{Q}_1 - A_1)} \right] \\
+ I_2 \left[ \Delta N_2(0)(\Delta T_1 + 2\tau') + A_2 \frac{(\Delta T_1 + 2\tau')^2}{2} \right]. \tag{13.26}
\]
Introducing the abbreviation
\[
E_1 = N_{1\text{max}} \tau' + \frac{(N_1)^2}{2(\bar{Q}_1 - A_1)}, \tag{13.27}
\]
and dividing Eq. (13.26) by \((\Delta T_1 + 2\tau')\) yields
\[
G_w(\Delta T_1 + 2\tau') = \frac{I_1 E_1}{\Delta T_1 + 2\tau'} + I_2 \left[ \Delta N_2(0) + A_2 \frac{\Delta T_1 + 2\tau'}{2} \right]. \tag{13.28}
\]
This expression shall be minimized under the constraint \(\Delta T_1 \geq T_i\). In order to determine the minimum, we set the derivative with respect to \(\Delta T_1\) to zero and get
\[
0 = \frac{dG_w(\Delta T_1 + 2\tau')}{d\Delta T_1} = -\frac{I_1 E_1}{(\Delta T_1 + 2\tau')^2} + \frac{I_2 A_2}{2} \cdot \tag{13.29}
\]
The minimum is located at
\[(\Delta T_1 + 2\tau')^2 = \frac{2I_1E_1}{I_2A_2}, \quad (13.30)\]

if
\[
(\Delta T_1 + 2\tau')^2 \geq (T_1 + 2\tau')^2 = \left(\frac{\Delta N_1^{\max}}{Q_1 - A_1} + 2\tau'\right)^2. \quad (13.31) \{\text{impli}\}
\]

Otherwise the minimum is on the boundary \(\Delta T_1 = T_1\).

**13.2.4 “Slower-is-Faster Effect” II**

Equation (13.31) implies the condition
\[
\left(\frac{I_1}{I_2A_2} - \frac{1}{Q_1 - A_1}\right) \frac{\Delta N_1^{\max}}{Q_1 - A_1} + 2\tau' \left(\frac{I_1}{I_2A_2} - \frac{2}{Q_1 - A_1}\right) \Delta N_1^{\max} \geq 4\tau'^2. \quad (13.32)
\]

As \(\tau'^2\) is very small, it is practically fulfilled when both terms in round brackets on the left-hand side are positive, i.e. if \(I_1(Q_1 - A_1) \geq 2I_2A_2\) or if
\[
I_1Q_1 - I_2A_2 \geq I_1A_1 + I_2A_2. \quad (13.33)
\]

**HIER NOCH ETWAS ERLÄUTERUNGEN ZUM EFFEKT!**

**13.3 Control Parameters and Phase Diagram of Traffic Signal Control**

In order to get an idea of the complexity of optimal traffic light control, let us ask about the dimension of the phase space. For reasons of simplicity, it is common to transform all parameters to dimensionless form. For example, it is reasonable to measure times in units of \(\tau'\). Moreover, most formulas can be expressed in terms of relative flows such as
\[
\frac{I_1A_1}{I_2A_2}, \frac{A_1}{Q_1}, \frac{A_2}{Q_2}. \quad (13.34)
\]

Note that parameters like
\[
\frac{I_1A_1}{I_2(Q_2 - A_2)} \quad \text{and} \quad \frac{I_2A_2}{I_1(Q_1 - A_1)} \quad (13.35)
\]
can be expressed through the previous set of parameters. While a single intersection with 2 phases only is characterized by 3 control parameters, the optimal operation of \(N\) intersections depends on \(3^N\) parameters. In view of this, it is obvious that the optimal coordination of many traffic lights constitutes a hard computational problem.

Here, we will focus on the different operation regimes for one phase only. Summarizing our findings in the previous two subsections and dividing by \(I_1A_1\) we find the following regimes:
1. From the point of view of minimizing the overall waiting times, it is not advisable to turn green if
\[
\frac{\hat{Q}_1}{A_1} - \frac{I_2 A_2}{I_1 A_1} \leq 1
\] (13.36)
In that situation we would have \( I_1 \hat{Q}_1 < I_2 A_2 \), i.e. there more vehicle arriving on road section 2 than what road section 1 can maximally serve. Therefore, it is better to keep up serving road section 2.

Of course, one may consider relaxing the principle of minimizing of overall waiting times in such cases, but this will make traffic operation less efficient. Alternatively, one may allow to turn on red, whenever there is a sufficient gap in the vehicle stream, so that no vehicle will be stuck behind a red light forever. It can have positive effects, however, to stop serving side roads during rush hours. Not only will this reduce disturbances in the service of the main roads by a temporary re-organization of traffic operation. In fact, the reduction in the number of signal phases can free up additional capacity (see Sec. ??). The increase of waiting times in the side roads will also motivate drivers to take the main roads, according to travel-time based route choice models (see Sec. ??). This adaptive driver behavior will re-inforce the re-organization of traffic light operation during rush hours.

2. In the case
\[
1 < \frac{\hat{Q}_1}{A_1} - \frac{I_2 A_2}{I_1 A_1} \leq 1 + \frac{I_2 A_2}{I_1 A_1},
\] (13.37)
the optimal duration of the green phase is \( \Delta T_i = T_i \), i.e. road section 1 is served until all vehicles in the queue have left the road section. This can be considered as the normal mode of operation.

3. However, if
\[
1 < \frac{\hat{Q}_1}{A_1} - \frac{I_2 A_2}{I_1 A_1} \leq 1 + \frac{I_2 A_2}{I_1 A_1},
\] (13.38)
it is advisable to have excess green times, which may be interpreted as a “slower-is-faster effect”. In this situation, it takes some additional time to accumulate enough vehicles on road section 2 to guarantee an efficient service in view of the inefficiencies by the switching times \( \tau' \).

13.3.1 Plausibility Considerations

Obviously, for constant arrival flows we do not have a situation in which it is advisable to interrupt the service of a vehicle queue before it is fully dissolved. This is plausible, as the interruption of service would cause a large additional waiting time for vehicles having to wait for another green time until they can leave the road section.

Let us make two further plausibility checks of our results. First, the case in which a road section is not being served, should only occur in the traffic
regime, where more vehicles arrive than the maximum intersection capacity can handle. This situation may definitely arise if

\[ I_1 A_1 + I_2 A_2 \geq I_1 \hat{Q}_1, \tag{13.39} \]

which is in agreement with Eq. (??).

Second, we should not have excess green times \( \Delta t_i > 0 \) when the arrival flows reach the intersection capacity. In fact, as \( (I_1 A_1 + I_2 A_2) \) approaches \( I_1 \hat{Q}_1 \) from below, \( (I_1 \hat{Q}_1 - I_2 A_2) \) approaches \( I_1 A_1 \) from above, which means that we automatically approach the boundary of the regime with \( \Delta T_i = T_i \), as expected. That is, providing excess green time will not lead to ever-growing vehicle queues.

### 13.4 Principle of Traffic Signal Operation

Let us determine the value of the goal function in the cases (i) and (ii) discussed above. In case (i), with \( \Delta T_1 = T_1 = \Delta N^{\text{max}}_1 = \Delta N_1(0) + A_1 \tau \) we find

\[
G_w(\Delta T_1 + 2\tau') = I_1 \left[ \Delta N_1(0) + \hat{Q}_1 \tau' - (\hat{Q}_1 - A_1) \frac{T_1 + 2\tau'}{2} \right] \\
+ I_2 \left[ \Delta N_2(0) + A_2 \frac{T_1 + 2\tau'}{2} \right] \\
= I_1 \frac{\Delta N_1(0) + A_1 \tau'}{2} \left[ \frac{1}{2} + \frac{I_2 A_2}{I_1 (\hat{Q}_1 - A_1)} \right] + I_2 [\Delta N_2(0) + A_2 \tau'] \\
< I_1 [\Delta N_1(0) + A_1 \tau'] + I_2 [\Delta N_2(0) + A_2 \tau'], \tag{13.40}
\]

where the inequality follows from \( I_1 (\hat{Q}_1 - A_1) > I_2 A_2 \) in case (i). This also implies

\[
\frac{1}{2} + \frac{I_2 A_2}{I_1 (\hat{Q}_1 - A_1)} < 1, \tag{13.42}
\]

i.e. the first square bracket in (13.40) is always smaller than the second one.

A similar observation can be made in case (ii). With Eq. (??) and \( \sqrt{1 + x} < 1 + x/2 \) we have
\[ G_w(\Delta T_1 + 2\tau') = \frac{I_1 E_1}{\sqrt{2I_2 A_2}} + I_2 \left[ \Delta N_2(0) + \frac{A_2}{2} \sqrt{\frac{2I_1 E_1}{I_2 A_2}} \right] \]
\[ = \sqrt{2I_1 E_1 I_2 A_2} + I_2 \Delta N_2(0) \]
\[ = \sqrt{2I_2 A_2 I_1 \left[ \Delta N_1^{\text{max},\tau'} + \frac{(\Delta N_1^{\text{max}})^2}{2(\tilde{Q}_1 - A_1)} \right]} + I_2 \Delta N_2(0) \]
\[ = I_1 \Delta N_1^{\text{max}} \sqrt{\frac{I_2 A_2}{I_1(\tilde{Q}_1 - A_1) \left( 1 + 2\tau'(\tilde{Q}_1 - A_1) \Delta N_1^{\text{max}} \right)}} + I_2 \Delta N_2(0) \]
\[ < \sqrt{\frac{I_2 A_2}{I_1(\tilde{Q}_1 - A_1)}} \left( I_1 \Delta N_1^{\text{max},\tau'} I_1(\tilde{Q}_1 - A_1) \right) + I_2 \Delta N_2(0) \]
\[ = \sqrt{\frac{I_2 A_2}{I_1(\tilde{Q}_1 - A_1)}} I_1 \Delta N_1^{\text{max}} \]
\[ + \sqrt{I_2 A_2 I_1(\tilde{Q}_1 - A_1)\tau'} + I_2 \Delta N_2(0) \].

As case (ii) implies \( I_1(\tilde{Q}_1 - A_1) > 2I_2 A_2 \) and \( \tau' \) is small compared to \( \Delta T_1 \), the goal function may be estimated by
\[ G_w(\Delta T_1 + 2\tau') < \frac{I_1 \Delta N_1^{\text{max}}}{\sqrt{2}} + \frac{I_1(\tilde{Q}_1 - A_1)}{\sqrt{2}} \tau' + I_2 \Delta N_2(0). \]

Again, as in Eq. (13.41), the prefactor of \( I_1 \Delta N_1^{\text{max}} = I_1[\Delta N_1(0) + A_1\tau'] \approx I_1 \Delta N_1(0) \) is smaller than 1, while the prefactor of \( I_2 \Delta N_2(0) \) is 1. This has important consequences: Let us consider for a moment to serve road section 2 another time rather than serving road section 1 after the amber time period.

The corresponding goal functions would be given by the above functions, but with interchanged indices 1 and 2. It turns out that, given \( I_2 \Delta N_2(0) = 0 \) and a large number \( I_1 \Delta N_1(0) \) of waiting vehicles, the smaller value of the goal function \( G_w \) is reached by serving road section 1, as assumed before. If the prefactor of \( I_1 \Delta N_1(0) \) in the above formulas was not smaller than 1, the situation would be different.

Hence, the principle of traffic signal operation could be formulated as follows:

1. At the beginning of an amber time period, determine the value of the goal function for each road section, assuming that it will be served next and up to the time point when the goal function starts to increase again (i.e. until the goal function reaches its first minimum).
2. Compare the values of the goal functions determined for the different road sections, which correspond to the respective average increase in the overall waiting time during the service period (and the amber time periods before and after).
3. After the amber time period, serve the road section \( j \) with the smallest value of the goal function next, until the end of the time period \( \Delta T_j \) is reached.

4. Then, start the same procedure again.

### 13.4.1 Effect of Vehicle Platoons on Traffic Signal Operation

The above considerations have been performed for the case of constant vehicle flows only. In the case of vehicle platoons, however, it may be advantageous to give priority to the platoon. That is, if the number of vehicles in the currently served queue is small, but stopping the platoon would stop a large number of vehicles, it would be better to interrupt the service in order to let the platoon pass. Such a prioritization would establish a *green wave*. However, how can we assess the relative disadvantage of stopping a platoon compared to stopping the service of a vehicle queue at the intersection? If the vehicle platoon is not dense enough, it would certainly be better to continue serving the queue until it is fully dissolved.

In order to solve this problem, we need to use the goal function \( G_a(t) \) defined in Eq. (??), which also takes into account the effect of stopping vehicles on the overall waiting times. If we repeat our above computations for the function \( G_a(t) \) rather than \( G_w(t) \), nothing changes in case (i). However, in case (ii) we find

\[
F_a(\Delta T_i + 2\tau') = F_w(\Delta T_i + 2\tau') + I_1 T_1' C_1 \rho_{\text{max}} (\Delta T_i + 2\tau' - \Delta t_1) + I_2 T_2' C_2 \rho_{\text{max}} (\Delta T_i + 2\tau'),
\]

which implies

\[
G_a(\Delta T_i + 2\tau') = G_w(\Delta T_i + 2\tau') + I_1 T_1' C_1 \rho_{\text{max}} + I_2 T_2' C_2 \rho_{\text{max}} - I_1 T_1' C_1 \rho_{\text{max}} \frac{\Delta T_i - T_i}{\Delta T_i + 2\tau'}.
\]

In cases where an excess green time is favorable, the corresponding formula for the green time duration is

\[
(\Delta T_i + 2\tau')^2 = \frac{2I_1}{I_2 A_2} [E_1 + T_1' C_1 \rho_{\text{max}} (T_i + 2\tau')] .
\]

In order to support excess green times, the condition \((\Delta T_i + 2\tau')^2 > (T_i + 2\tau')^2\) must again be fulfilled, which requires

\[
4\tau'^2 \leq \left( \frac{I_1}{I_2 A_2} - \frac{1}{Q_1 - A_1} \right) \frac{\Delta N_{\text{max}}}{Q_1 - A_1} + 2\tau' \left( \frac{I_1}{I_2 A_2} - \frac{2}{Q_1 - A_1} \right) \Delta N_{\text{max}}
+ \frac{2I_1 (Q_1 - A_1)}{I_2 A_2} T_1' C_1 \rho_{\text{max}} \left( \Delta N_{\text{max}} + \frac{2\tau}{Q_1 - A_1} \right).
\]

Under the condition

\[
\frac{I_1 T_1' C_1 \rho_{\text{max}}}{I_2 A_2} > 2\tau ,
\]

(13.49)
the non-negativity of the first round bracket in condition (13.48) will usually be sufficient to satisfy it. This implies that the boundary for the possibility of excess green times $\Delta t_1 > 0$ is shifted from $(I_1\hat{Q}_1 - I_2A_2) > I_1 A_1 + I_2 A_2$ to $(I_1\hat{Q}_1 - I_2A_2) > I_1 A_1$, if condition (13.49) is fulfilled. In other words, if the inequality (13.49) is fulfilled, the service of platoons introduces excess green times $\Delta t_i > 0$ in the operation regime where we had $\Delta t_i = 0$ without the consideration of platoons. This is in agreement with our expectations.

The inequality (13.49) also provides a quantitative criterium for the values $I_1C_1\rho_{\text{max}}/I_2A_2$ for which stopping the service of a vehicle queue at the intersection should be considered in favor of giving way to a platoon. We see that high values of $C_1 = [1/A_1 - 1/(\rho_{\text{max}}V_0^0)]^{-1}$ (and, hence, $A_1$) and low values of $A_2$ support the prioritization of platoons, which is plausible. Note that, the consideration of the effect of stopping vehicle platoons (i.e. $T'_1 > 0$) does not introduce new operation regimes. It just shifts the boundaries of the operation regimes listed in Sec. ??.

### 13.4.2 Interrupted Inflow or Outflow

Of course, the above described traffic light control principle cannot only be applied to constant arrival flows. We expect that the traffic light would normally turn red when there is no inflow anymore (which means $A_1 = 0$ and $C_1 = 0$) or when the outflow is blocked due to a totally congested road section ($\hat{Q}_1 - A_1 = 0$). Studying the time-dependence of the function $G(t)$ shows that, in any of these two cases, the function $G(t)$ will start to increase again, which means that the green time will be interrupted before according to the minimization of the goal function $G(t)$. This shows that our traffic light control principle behaves reasonably.

### 13.4.3 More than Two Traffic Phases

Let us now discuss, how our control principle can be extended to deal with more than two traffic phases. First or all, we should state that the optimize-one-phase approach is immediately applicable to an arbitrary number of phases. After the completion of one green phase and the subsequent switching time, it will compare the minimal $G(t)$ values for all possible green phases and decide for the green phase which reaches the lowest value. This green phase will be operated up to the time at which the minimum of $G(t)$ is reached.

The procedure for the optimize-one-cycle approach is more difficult. Note that the above approach can be easily extended to consider more subsequent phases in the optimization process, and that the balance principle (??) will apply for all subsequent phases. The length of one cycle in terms of the number of phases it includes may, however, vary considerably. For example, in principle two of the three or more possible phases may be repeated several times, before another phase is served again. Therefore, one should rather define a cycle by the shortest sequence until one phase reappears. Nevertheless,
we point out that the number of possible sequences (combinations) of phases grows exponentially with the length of a cycle. Therefore, we can already see that the optimization problem becomes a NP-hard, combinatorial optimization problem when the phases of several neighboring traffic lights shall be optimized together, since the optimum control of a traffic light depends on its neighbors. So, there is a complex mutual interdependence of the traffic flows and waiting times of vehicles.

13.4.4 Limited Forecast Time Horizon

In some sense, the traffic light optimization simplifies due to the limited time horizon, over which we can make sufficiently reliable traffic forecasts, which are needed for the optimization of the goal function. Therefore, one can restrict the traffic light optimization to time periods $1/\lambda$ over which the traffic forecast can be done with sufficient accuracy. This suggests to replace the optimization of one or several full cycles by the optimization of a few subsequent phases. For example, the above optimize-one-cycle approach can be easily generalized to a optimize-two-phases approach, by considering all possible combinations of two subsequent phases and selecting those two phases which yield the minimum value of the goal function $G(t)$ until the end of the second green phase plus the following setup time. One could also demand that the number of subsequent phases we are optimizing over should cover a traffic light operation longer than the time interval $1/\lambda$. This time interval, however, is expected to be shorter, the more switches of traffic lights are made. As a consequence, optimizing over a fixed number of phases will be a reasonable approach, and optimizing over two or three phases is already expected to yield good results. Note, however, that the considerations for stationary cycles in Sec. ?? cannot be transferred anymore to cases of three or more traffic phases.

If the optimization of the goal function is anyway done numerically (e.g. with a gradient method), one may also consider to use the exponentially weighted waiting time approach discussed in Sec. 13.1.3. Such an approach fits well to the formulation of economic problems considering future developments (e.g. future returns). This exponentially weighted approach can be still written up in a closed form for piecewise constant arrival flows, as shown in Appendix ?? . It leads to implicit, transcendent(al) equations that cannot solved analytically anymore, but they can be easily solved with standard numerical solution methods.

13.5 Coordination between Several Traffic Lights??

One of the important questions of optimal traffic light control is whether a good coordination between neighboring traffic lights can be reached or whether every traffic light is optimizing green phases in a selfish way. In the latter case, we may get stuck in a user equilibrium or local optimum, while
suitable coordination is needed to reach the system optimum (or at least get close to it). In the following, we will assume piecewise constant inflows, since step functions are a reasonable approach to vehicle flows in signal-controlled traffic systems. Moreover, we will minimize the goal function $G$ for several neighboring intersections $j$, i.e. we will minimize the overall delay and waiting times of several intersections.

13.5.1 Synchronization with an Upstream Traffic Light

Let us assume two subsequent intersections 1 and 2. Each intersection may have two unidirectional inflows and two unidirectional outflows. The sideways inflow towards section 1 will be denoted by the index 0, while the sideways outflow from section 2 will be denoted by the index 4 (see Fig. ??). For simplicity we will assume that all road sections are so long that they are never fully congested. DER GEGENTEILIGE FALL WAERE ABER AUCH SEHR INTERESSANT ZU UNTERSUCHEN!

The number $I_i \Delta N_i(t)$ of vehicles in the queue on road section $i$ changes in time according to

$$I_i \frac{d\Delta N_i}{dt} = I_i A_i(t - L_i/V_0) - I_i O_i(t)$$

with

$$I_i O_i(t) = \gamma_i(t) \begin{cases} I_i \hat{Q} & \text{if } \Delta N_i(t) > 0, \\ I_i A_i(t - L_i/V_0) & \text{otherwise}, \end{cases}$$

where $\gamma_i(t) = 1$ if the traffic light is green for road section $i$, otherwise $\gamma_i(t) = 0$. $L_i$ is the length of the road section, $V_0$ the speed limit, and $\hat{Q}$ the maximum outflow from a queue. For simplicity, the inflows $A_0$, $A_1$, and $A_3$ are assumed to be constant.

For intersection 1, we will assume a periodic fixed time control with a green phase of duration $\Delta t_0$ for flow 0, starting at time $t = 0$. The green phase for flow 1 starts at $\Delta t_0 + \tau$ and lasts for a time period $\Delta t_1$. The cycle time is $T_1^{cy} = \Delta t_0 + \Delta t_1 + 2\tau$. With $k \in \{0, 1, 2, \ldots \}$, $T_0 = A_0(\Delta t_1 + 2\tau)/E_0$, and $T_1 = A_1(\Delta t_0 + 2\tau)/(\hat{Q}_1 - A_1)$, the outflows can be written as follows:

$$I_0 O_0(t) = \begin{cases} I_0 \hat{Q} & \text{if } kT_1^{cy} \leq t < kT_1^{cy} + T_0, \\ I_0 A_0 & \text{if } kT_1^{cy} + T_0 \leq t < kT_1^{cy} + \Delta t_0, \\ 0 & \text{otherwise} \end{cases}$$

and

$$I_1 O_1(t) = \begin{cases} I_1 \hat{Q} & \text{if } kT_1^{cy} + \Delta t_0 + \tau \leq t < kT_1^{cy} + \Delta t_0 + \tau + T_1, \\ I_1 A_1 & \text{if } kT_1^{cy} + \Delta t_0 + \tau + T_1 \leq t < kT_1^{cy} + \Delta t_0 + \tau + \Delta t_1, \\ 0 & \text{otherwise.} \end{cases}$$
This implies

\[
I_2 A_2(t) = \alpha_{20} I_0 O_0(t) + \alpha_{21} I_1 O_1(t)
\]

\[
= \begin{cases} 
\alpha_{20} I_0 \hat{Q} & \text{if } k T_2^{\text{cyc}} \leq t < k T_1^{\text{cyc}} + T_0, \\
\alpha_{20} I_0 A_0 & \text{if } k T_1^{\text{cyc}} + T_0 \leq t < k T_1^{\text{cyc}} + \Delta t_0, \\
\alpha_{21} I_1 \hat{Q} & \text{if } k T_1^{\text{cyc}} + \Delta t_0 + \tau \leq t < k T_1^{\text{cyc}} \\
& + \Delta t_0 + \tau + T_1, \\
\alpha_{21} I_1 A_1 & \text{if } k T_1^{\text{cyc}} + \Delta t_0 + \tau + T_1 \leq t < k T_1^{\text{cyc}} + \Delta t_0 + \tau + \Delta t_1, \\
0 & \text{otherwise.} 
\end{cases}
\]  

(13.54)

It is obvious that our traffic light optimization algorithm will eventually produce a period traffic light operation at intersection 2. If we have a periodic operation of the traffic light with two phases per period, we can assume green time periods of \( \Delta t_2 \) and \( \Delta t_3 \) for flows 2 and 3, a cycle time of \( T_2^{\text{cyc}} = \Delta t_2 + \Delta t_3 + 2\tau \), and an offset of \( \Delta t \). That is

\[
\gamma_2(t) = \begin{cases} 
1 & \text{if } k T_2^{\text{cyc}} + \Delta t \leq t < k T_2^{\text{cyc}} + \Delta t + \Delta t_2, \\
0 & \text{otherwise},
\end{cases}
\]

(13.55)

and

\[
I_2 O_2(t) = \gamma_2(t) \begin{cases} 
I_2 \hat{Q} & \text{if } \Delta N_2(t) > 0, \\
I_2 A_2(t - L_2/V_0) & \text{if } \Delta N_2(t) = 0.
\end{cases}
\]

(13.56)

Furthermore, with \( T_3 = A_3 \cdot (\Delta t_2 + 2\tau)/E_3 \) we have

\[
I_1 O_3(t) = \begin{cases} 
I_3 \hat{Q} & \text{if } k T_2^{\text{cyc}} + \Delta t + \Delta t_2 + \tau \leq t < k T_2^{\text{cyc}} + \Delta t + \Delta t_2 + \tau + T_3, \\
I_3 A_3 & \text{if } k T_2^{\text{cyc}} + \Delta t + \Delta t_2 + \tau + T_3 \leq t < k T_2^{\text{cyc}} + \Delta t + \Delta t_2 + \tau + \Delta t_3, \\
0 & \text{otherwise.}
\end{cases}
\]

(13.57)

With these specifications, we can now minimize the goal function, varying the green time periods \( \Delta t_2 \), \( \Delta t_3 \), and the offset \( \Delta t \). As the delay and waiting times at intersection 1 are not influenced by the traffic light optimization at intersection 2, we can focus on the ones accumulating on road sections 2 and 3. For road section 3, with \( t_k := k T_2^{\text{cyc}} + \Delta t + \Delta t_2 + \tau \) we find

\[
\Delta N_3(t) = \begin{cases} 
A_3 \cdot (\Delta t_2 + 2\tau) & \text{if } t_k \leq t < t_k + T_3, \\
-E_3 (t - t_1) & \text{if } t_k + T_3 \leq t < t_k + \Delta t_3, \\
0 & \text{otherwise.}
\end{cases}
\]

(13.58)

Writing up the goal function \( G(t) \) explicitly turns out to be quite cumbersome. However, it is quite clear that the optimal cycle time \( T_2^{\text{cyc}} \) would be a fraction or multiple of \( T_1^{\text{cyc}} \). In the following, we will assume \( T_2^{\text{cyc}} = T_1^{\text{cyc}} \) for simplicity. Then, we can only vary the green time period \( \Delta t_2 \) and the offset \( \Delta t \).
13.5 Coordination between Several Traffic Lights

During the cycle time $T^{\text{cy}c}_2 = T^{\text{cy}c}_1$, we will have the arrival rate $\alpha_{20} \hat{Q}$ for the time period $T_0 = A_0 \cdot (\Delta t_1 + 2\tau)$, the arrival rate $\alpha_{20} A_0$ for the time period $\Delta t_0 - T_0$, then no flow for the setup time $\tau$, afterwards the arrival rate $\alpha_{21} \hat{Q}$ for the time period $T_1 = A_1 \cdot (\Delta t_0 + 2\tau)$, the arrival rate $\alpha_{21} A_1$ for the time period $\Delta t_1 - T_1$, and again no flow for the setup time $\tau$. It is obvious that it is favorable to allow the larger platoon of vehicles to pass intersection 2 without stopping. For example, assume that $\alpha_{20} I_0 \hat{Q} T_0 > \alpha_{21} I_1 \hat{Q} T_1$. Then, the last vehicle of the platoon of vehicles originating from road section 0 should pass intersection 2 immediately before the traffic light turns red. (This is because, afterwards no vehicle would be stopped for the time period $\tau$.) As all previously arrived vehicles stemming from road section 1 must be served before the arrival of the first vehicle of that platoon, the traffic light on road section 2 needs to turn to green a time period

$$\Delta T_1 := \alpha_{21} I_1 A_1 T^{\text{cy}c}_1 / (I_2 (\hat{Q}_2 - A_2))$$  \hspace{1cm} (13.59)

before the arrival of the first vehicle of the platoon from road section 0. This establishes the synchronization of traffic lights. Specifically, the green time must be $\Delta t_2 = \Delta T_0 + \Delta T_1$ with $\Delta T_0 := \alpha_{20} I_0 A_1 T^{\text{cy}c}_0 / (I_2 (\hat{Q}_2 - A_2))$, PRUEFEN (ICH GLAUBE, DASS $\Delta t_2$ GROESSER IST, WENN $A_3$ KLEIN IST) and it must start a time period $\Delta T_1 - L_2 / V_0$ before the first vehicle of platoon 0 leaves road section 0, which means

$$\Delta t = L_2 / V_0 - \Delta T_1.$$  \hspace{1cm} (13.60)

PRUEFEN! Note that this can be positive or negative, depending on the respective values of $L_2$ .......... and the number of vehicles queued on road section 2. Moreover, a change in the turning fractions .......... can change the optimal time shift.

ES WAERE SPANNEND, DIE VERSCHIEDENEN MOEGLICHKEITEN EINMAL NUMERISCH AUSZULOTEN. INSBSONDERE WAERE ES INTERESSANT, AUCH EINEN FALL ZU KONSTRUIEREN, IN DEM EIN SERVICE DER PERIODE 2 AUFTRITT, D.H. WO RICHTUNGEN 2 und 3 ZWEIMAL BEDIEN WERDEN, VIelleicht SOGAR EINMAL LANG UND EINMAL KURZ, BEVOR SICH ALLES WIEDERHOLT.

13.5.2 Synchronization with a Downstream Traffic Light

The synchronization with a fixed time schedule of a downstream traffic light can be treated analogously. The only difference is that, rather than varying the green time periods $\Delta t_2$, $\Delta t_3$, and the offset $\Delta$, one minimized the goal function $G(t)$ under variation of the green times $\Delta t_0$, $\Delta t_1$, and the offset $\Delta$. 
13.5.3 Synchronization with Several Traffic Lights

13.5.4 Different Operation Modes

Synchronization may not necessarily be a good strategy. This rather depends on the flow conditions. In fact, one needs to distinguish several operation regimes, in which different optimization strategies apply:

I. In the free-flow, low-utilization regime, demand is considerably below capacity. From the point of view of minimizing production times (not considering setup costs, here), the application of a first-in-first-out principle would perform best. Individual jobs would be served just upon their arrival. Moreover, a large spectrum of different products, i.e. a large variety can be produced. However, the overall throughput is low due to the small order flow.

II. If the order flow gets higher, one enters the mutually obstructed, platoon-formation regime. That is, conflicts of usage are likely, and waiting times are unavoidable. Queues are forming in the system, but they actually help to reach a higher throughput: It is more efficient to serve educts already waiting to be processed rather than having to wait for their arrival. A minimization of waiting and production times leads to the following strategy: Longer queues should be prioritized compared to shorter ones. However, interrupting a smoothly progressing production process is unfavorable. Therefore, dense moving platoons are prioritized compared to short enough standing ones. This can lead to a synchronization of successive machines or service stations (but with a transportation-time related phase shift). In traffic systems, such operation modes are called “green waves”.

III. When the demand exceeds capacity, one faces the congested, queue-dominated regime. This should be avoided by enforcing that jobs enter the production system with suitable delays. Otherwise, queues tend to form everywhere. A promising optimization strategy in such cases is the application of a flow bundling principle. This basically means to reduce the variety of the products in order to minimize conflicts of usage and the heterogeneity in the system. Thereby, it is possible to increase the overall throughput of the system even under conditions of high utilization.

IV. At extreme utilizations, when the demand is considerably above capacity, one will end up in the heavily congested, gap propagation regime. In that case, almost all the storage capacity is used up everywhere in the system. Now, the focus is on uninterrupted gap propagation, i.e. “green waves” for gaps. Larger gaps should be given priority compared to smaller gaps.

The above list of operation regimes is not meant to be complete. It should rather indicate that there is not a single, simple strategy to optimize the operation of a production system. Note that optimization is not only needed within each operation regime. Another challenge is to manage optimal transitions between these regimes. It is also difficult to define exactly, under which
conditions these transitions take place. As a consequence, there are currently more questions than answers. However, based on their special methodological knowledge, physicists are expected to make a lot of application-oriented, but fundamental contributions, here.

13.5.5 Chaotic Control of Three Intersecting Roads

13.6 A Fundamental Diagram for Urban Traffic Flow?

Many traffic models for freeways or urban roads assume a flow-density relationship called the fundamental diagram (see Chap. ...). The concept of the fundamental diagram has been quite useful for the understanding of capacity effects, different traffic states (see Sec. ...), and the traffic dynamics, in particularly the propagation of shock fronts (see Sec. ...). It is, therefore, not surprising that people have tried to find fundamental relationships for urban traffic as well. Robert Herman and Ilya Prigogine have addressed this issue early on with a “two-fluid approach” [?], considering moving and standing traffic and their interrelationships. Recently, Carlos Daganzo and coauthors have continued this analysis [?] and presented empirical evidence [?]. In the following, we will determine a formula for the average waiting time of vehicles and derive from it a relationship for the average speed, assuming a simple scenario.

Specifically, let us focus on a single intersection with constant inflows $A_i$. Furthermore, let us assume to operate the traffic lights periodically in a way that clears the buffers and does not provide excess green times. For a two-phase operation, the cycle time is then given by Eq. (13.10) 

$$T_{cyc} = \frac{\sum_i \tau'_i}{1 - \sum_j A_j/Q_j},$$

(13.61)

where $\tau'_i$ represent the respective switching times (including clearing times).

According to Eq. (13.10), the maximum number of queued vehicles is given by $\Delta N^\text{max}_i = A_i t_{0i}$, where

$$t_{0i} = (1 - u_i) T_{cyc} = (1 - u_i) \frac{\sum_i \tau'_i}{1 - \sum_j A_j/Q_j}$$

(13.62)

is the overall red time for the vehicle flow $i$, and

$$u_i = \tau_i = \int_{-\infty}^{t} dt' \lambda e^{-\lambda'(t-t')} \gamma(t')$$

(13.63) \{gammaweight\}

represents the average green time fraction for vehicle flow $i$. Following Eq. (13.12) with $O_i = Q_i$, the green time required to resolve the queue in traffic flow $i$ is
This implies
\[ u_i = \frac{A_i}{Q_i}, \] (13.65)
so that \( u_i \) can be interpreted as utilization of the road section used by flow \( i \).

In order to establish a fundamental relationship for city traffic, a similar exponentially weighted averaging as in Eq. (??) is required for other time-dependent traffic variables as well (see also Sec. ?? on this subject). For example, the average waiting time of vehicles is given by the arithmetic average of the waiting time of the first vehicle in the queue and the last one, as vehicles in our example are arriving with equal separation in time. While the waiting time of the last vehicle is zero, the waiting time of the first one is \( t_{0i} \). Therefore, the average waiting time of vehicles of flow \( i \) is \( t_{0i}/2 \). Therefore, the average travel time is the travel time \( L_i/V_{0i} \) in free traffic plus the average waiting time
\[ t_{0i}/2 = \frac{\Delta N_i^{\text{max}}}{2A_i} = \frac{\Delta N_i}{A_i}. \] (13.66)

Here, we have taken into account that the vehicle queue grows and shrinks linearly, so it must be given have an average length of \( \Delta N_i = \Delta N_i^{\text{max}}/2 \).

As a consequence of Eq. (13.66), the average velocity to pass the link of length \( L_i \) is
\[ V_i = \frac{L_i}{V_i^0 + \frac{t_{0i}}{2}} = \left( \frac{1}{V_i^0} + \frac{\bar{\rho}_i}{A_i} \right)^{-1} = \left( \frac{1}{V_i^0} + \frac{\bar{\rho}_i}{u_i Q_i} \right)^{-1}, \] (13.67)
where
\[ \bar{\rho}_i = \frac{\Delta N_i}{L_i} \] (13.68)
represents the density of vehicles on the road section used by flow \( i \). The density can also expressed by other variables, as follows:
\[ \bar{\rho}_i = \frac{\Delta N_i^{\text{max}}}{2L_i} = \frac{A_i t_{0i}}{2L_i} = \frac{A_i (1 - u_i) T_{\text{cyc}}}{2L_i} = \frac{A_i \sum t'_i (1 - u_i)}{(1 - \sum u_j)}. \] (13.69)

Therefore, remembering Eq. (13.67), we can express the average velocity via
\[ T_i(u_i) = \frac{L_i}{V_i} = \frac{L_i}{V_i^0} + \frac{(1 - u_i)}{(1 - \sum u_j)} \sum t'_i. \] (13.70)

This means, one can either express the average velocity as a function of the average density \( \bar{\rho}_i \) and the average arrival rate \( A_i \), or by the utilizations \( u_i \), which is more simple. Note that both, Eq. (13.67) and (13.70) establish fundamental
relationships for urban traffic, which may be compared to the phenomenological formula

$$T_i(A_i) = T_0 \left\{ a' \left[ 1 + \left( \frac{A_i}{\hat{Q}_i} \right)^{b'} \right] \right\}, \tag{13.71}$$

where $a'$ and $b'$ are fit parameters \cite{[link?]}. (As this formula is often applied to values $A_i > \hat{Q}_i$ as well, it is theoretically not consistent. In other words, $\hat{Q}_i$ must be considered as another fit parameter, here, rather than the maximum flow.)

It should be stressed, that the above formulas are valid exactly only for the simple scenario we have been studying. For example, if the traffic light control provides excess green time, the cycle time is increased by a factor $c' > 1$, and the formulas for the average travel time $T_i$ and the average velocity $V_i$ become

$$T_i(u_i) = \frac{L_i}{V_i} = \frac{L_i}{V_i} + \frac{c'\pi_i L_i}{A_i} = \frac{L_i}{V_i} + \frac{c'\pi_i L_i}{\hat{Q}_i} = \frac{L_i}{V_i} + \frac{c'(1 - u_i)}{(1 - \sum_j u_j)} \frac{\sum_j t'_j}{2}. \tag{13.72}$$

Moreover, if the vehicle queue is not cleared during one green phase completely, the travel times increase further and the formulas look somewhat different (see Exercise \ldots). Average travel times are reduced, however, if traffic lights are coordinated, e.g. if they are operated to support green waves so that large vehicle platoons are served upon arrival at the traffic light. That is, the fundamental relationships for cities depend on details of the traffic light control, and the above formulas may be only approximately valid or may even have to be replaced by other ones for more realistic scenarios. But still, it can be considered to be a success to have found simple relationships at all. Like in classical queuing theory \cite{[link?]}, the formulas only depend on the utilizations $u_i = A_i/\hat{Q}_i$ and on the service characteristics.
14.1 “Bullwhip Effect” and “Business Cycles”

Let us now turn from traffic flows in one dimension to material flows in networks. We will treat this case in considerable detail, as the dynamics in networks is not only very important in order to understand real systems, but also very interesting. Our discussion will be carried out for the example of the dynamic input-output model proposed in Sec. ??.

Our computer simulations of the dis-aggregate (i.e. sector-wise) model of macroeconomic dynamics typically show asynchronous oscillations, which seems to be characteristic for economic systems. Due to phase shifts between sectoral oscillations, the aggregate behavior displays slow variations of small amplitude (see Fig. 14.1). If the function $L_i(P_i)$ and the parameters $\nu_i/\mu_i^2$ are suitably specified, the non-linearities in Eqs. (11.32) to (11.31) will additionally limit the oscillation amplitudes.

Such a business cycle theory differs from the most common ones by explaining the empirically observed irregular, i.e. non-periodic oscillations in a natural way (see Fig. 14.1). For example, $w$-shaped oscillations result as superposition of the asynchronous oscillations in the different economic sectors, while other theories have to explain this observation by assuming external perturbations (e.g. by “shocks” due to technological innovations). We will see that these oscillations result from the network structure of commodity flows (see Fig. ??) in combination with decentralized adjustments of the production rates by the different economic sectors.

The possible dynamic behaviors of the resulting dis-aggregate macroeconomic model can be studied by analytical investigation of limiting cases [?] and by means of a linear stability analysis around the equilibrium state characterized by

$$N_j(t) = N_j^0, \quad Y_j(t) = Y_j^0, \quad P_j(t) = P_j^0,$$

and

$$Q_j^0 - \sum_k c_{jk} Q_k^0 = Y_j^0 \quad \text{for all} \ j \in \{1, \ldots, m\}.$$
Fig. 14.1. Typical simulation result of the time-dependent gross domestic product $\sum_j Q_j(t)P_j(t)$ in percent, i.e. relative to the initial value. The input matrix was chosen as in Figs. 11.1 and 14.4a–d, but $Y_j^0$ was determined from averaged input-output data. $Q_j^0$ was obtained from the equilibrium condition, and the fluctuations $\xi_j(t)$ were specified as a Gaussian white noise with mean value 0 and variance $\sigma = 10,000$ (about 10% of the average final consumption). The initial prices $P_j(0)$ were selected from the interval $[0.9;1.1]$. Moreover, in this example we have assumed $L_j(P_i) = \max[0, 1 - 10(P_j - P_j^0)]$ and the parameters $\nu_j = 0.1, \mu_j = 0.0001, \hat{\alpha}_j = 1 = P_j^0, N_j^0 = Y_j^0$. Although this implies a growth of small oscillations (cf. Fig. 1d), the oscillation amplitudes are rather limited. This is due to the non-linearity of model equations (11.32) to (11.31) and due to the phase shifts between oscillations of different economic sectors $j$. Note that irregular oscillations with frequencies between 4 to 6 years and amplitudes of about 2.3% are qualitatively well compatible with empirical business cycles. Our material flow model can explain $w$-shaped, non-periodic oscillations without having to assume technological shocks or externally induced perturbations. The long-term growth of national economies was intentionally not included in the model in order to separate this effect from network-induced instability effects. (After Ref.[?])

The linearized equations for small deviations

\[
n_j(t) = N_j(t) - N_j^0, \quad p_j(t) = P_j(t) - P_j^0, \quad q_j(t) = Q_j(t) - Q_j^0 \quad (14.3)
\]

from the equilibrium state read
Fig. 14.2. The displayed phase diagram shows which dynamic behavior is expected by our dis-aggregate model of macroeconomic dynamics depending on the respective parameter combinations. The individual curves for different countries are a result of the different structures of their respective input matrices $C$. As a consequence, structural policies can influence the stability and dynamics of economic systems.

\[
\frac{dn_j}{dt} = q_j - \sum_k c_{jk} q_k + Y_j^0 |L_j^0(P_j^0)| p_j - \xi_j(t), \quad (14.4)
\]
\[
\frac{dp_j}{dt} = \frac{P_j^0}{N_j^0} \left( -\nu_j n_j - \mu_j \frac{dn_j}{dt} \right), \quad (14.5)
\]
\[
\frac{dq_j}{dt} = \frac{\hat{\alpha}_j Q_j^0}{N_j^0} \left( -\nu_j n_j - \mu_j \frac{dn_j}{dt} \right). \quad (14.6)
\]

Here, we have assumed the ratio

\[
\hat{\alpha}_j = \frac{\dot{\nu}_j}{\nu_j} = \frac{\dot{\mu}_j}{\mu_j} \quad (14.7)
\]

between the adjustment rates of the production rate and the price in sector $j$, and (for simplicity) the same ratio for the responsiveness parameters.
14 Oscillatory Network Flows

14.1.1 Solution for a Sequential Supply Chain

Let us first study the case \( c_{jk} = \delta_{j+1,k} \) of a sequential supply chain. In this case, the linearized equation for the inventories simplify to

\[
\frac{dn_j}{dt} = q_j(t) - q_{j+1}(t)
\]  \hfill (14.8) \{lin2\}

for \( j \in \{1, \ldots, m\} \) and

\[
\frac{dn_m}{dt} = q_m(t) - \frac{Y_0 f'(P_m^0)p_m(t) - \xi_m(t)}{=q_{m+1}(t)=y(t)}.
\]  \hfill (14.9)

Assuming a certain consumption behavior \( y(t) \), Eq. (14.8) will now be solved together with the Eq. (14.6), which describes the time-dependence of the production rate’s deviation from the equilibrium production \( Q_0^m \). Introducing the abbreviations

\[
\gamma_i = \frac{\hat{\alpha}_i \mu_j Q_j^0}{2N_j^0} > 0 \quad \text{and} \quad \omega_i^2 = \frac{\hat{\alpha}_i \nu_j Q_j^0}{N_j^0},
\]  \hfill (14.10)

deriving Eq. (14.6) with respect to \( t \) and inserting Eq. (14.8) results in the following set of second-order differential equations:

\[
\left\{ \begin{array}{l}
\frac{d^2q_j}{dt^2} + 2\gamma_j \frac{dq_j}{dt} + \omega_j^2 q_j(t) = f_j(t) \\
\end{array} \right. \text{for } j = \{1, \ldots, m\}. \hfill (14.11)
\]

This corresponds to the differential equation of a damped harmonic oscillator with damping constant \( \gamma \), eigenfrequency \( \omega \), and driving term \( f_j(t) \). The 2m eigenvalues of these equations are

\[
\tilde{\lambda}_{j,\pm} = -\gamma_j \pm \sqrt{\gamma_j^2 - \omega_j^2}.
\]  \hfill (14.12)

For \( \gamma_j \geq \omega_j \), the eigenvalues are always real and negative, while for \( \gamma_i < \omega \) we can write

\[
\tilde{\lambda}_{j,\pm} = -\gamma_j \pm i\sqrt{\omega_j^2 - \gamma_j^2}.
\]  \hfill (14.13)

In this case, the real value is \(-\gamma_j\), i.e. negative as well, but with an additional imaginary part which causes oscillations. That is, in all possible cases, we have exponentially damped, stable behavior in time. Nevertheless, in Sec. 14.1.2 we will find a convective instability, i.e. the growth of the oscillation amplitude from one supplier to the next one upstream.

Assuming periodic oscillations of the form \( f_j(t) = f_0^j \cos(\alpha t) \), the general solution of Eq. (14.11) is of the form

\[
q_j(t) = f_0^j F_j \cos(\alpha t + \varphi_j) + D_0^j e^{-\gamma_j t} \cos(\Omega_j t + \theta_j),
\]  \hfill (14.14)
where the parameters $D_j^0$ and $\theta_j$ depend on the initial conditions. The other parameters are given by
\[ \tan \varphi_j = \frac{2\gamma_j \alpha}{\alpha^2 - \omega_j^2}, \]  
(14.15)\[ \Omega_i = \sqrt{\omega_i^2 - \gamma_i^2}, \]  
(14.16)\[ F_j = \frac{1}{\sqrt{(\alpha^2 - \omega_j^2)^2 + 4\gamma_j^2 \alpha^2}}. \]  
(14.17)The dependence on the eigenfrequency $\omega_j$ is important for understanding the occurring resonance effect, which is particularly likely to appear, if the oscillation frequency $\alpha$ of the consumption rate is close to one of the resonance frequencies $\omega_j$. After a transient time much longer than $1/\gamma_j$ we find
\[ q_j(t) = f_0^j F_j \cos(\alpha t + \varphi_j). \]  
(14.18)
Equations (14.11) and (14.18) imply
\[ f_{j-1}^0 = f_j^0 \left[ \sqrt{\omega_j^2 \cos(\alpha t + \varphi_j) - 2\alpha \gamma_j \sin(\alpha t + \varphi_j)} \right] \]  
(14.19)\[ \tan \delta_j = \frac{2\alpha \gamma_j}{\omega_j^2}. \]  
PRUEFEN!!! and \[ f_{i-1}^0 = f_i^0 \sqrt{\omega_j^2 + 4\alpha^2 \gamma_j^2}. \]  
(14.20)Therefore, the set of equations (14.11) can be solved successively, starting with $j = m$ and progressing to lower values of $j$.

14.1.2 Convective Instability due to Resonance: The “Bullwhip Effect”

Let us analyze the relationship (14.20) for the oscillation amplitudes more closely. They would increase from one supplier to the next upstream one, if
\[ \frac{f_{i-1}^0}{f_i^0} = \left( 1 + \frac{\alpha^4 - 2\alpha^2 \omega_j^2}{\omega_j^4 + 4\alpha^2 \gamma_j^2} \right)^{-1/2} > 1. \]  
PRUEFEN!!!
One can see that this resonance effect can occur for $0 < \alpha^2 < 2\omega_j^2$. PRUEFEN!!!
Therefore, variations in the consumption rate are magnified whenever the sensitivity ?????????? $\dot{\nu}_j = \dot{\alpha} \nu_j$ towards deviations from the optimum inventory is too large. Surprisingly, as the frequency $\alpha$ at which the consumption rate changes may have any value, the attempt to maintain a constant inventory $N_i(t) = N_0^i$, will always lead to an increase of the inventory from one supplier to the next one upstream. This increase is known as “bull-whip effect”. Note that the bull-whip effect occurs despite the negative eigenvalues (?), which implies stable amplitudes in time. However, it is still possible to have a so-called convective instability, i.e. an increase in the amplitude from one supplier to the next, which is due to the resonance effect (see Fig. ??).
For small enough $\gamma$, corresponding to $\frac{\omega_j^2}{\omega_i^2}$, a resonance effect with an amplification factor greater than 1 can be observed. Perturbations with a frequency $\omega$ close to the eigenfrequencies $\omega_j$ are amplified and cause variations in the inventories to grow along the supply chain. This is responsible for the “bullwhip effect”.

14.1.3 Stability Analysis for Commodity Flows in Networks

Let us now come back to the linearized equations (14.4) to (14.6) of the general dynamic input-output model. Summarizing the variables by vectors $n = (n_1, \ldots, n_m)'$, $p = (p_1, \ldots, p_m)'$, and $q = (q_1, \ldots, q_m)'$, we can write the linear system of ordinary differential equations in matrix representation

$$\frac{d}{dt} \begin{pmatrix} n(t) \\ p(t) \\ q(t) \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \begin{pmatrix} n(t) \\ p(t) \\ q(t) \end{pmatrix}.$$  \hspace{1cm} (14.22)

Herein, $A_{il} = (a_{jk}^{il})$ are $m \times m$ matrices, whose entries $a_{jk}^{il}$ are defined as follows:

$$a_{1k}^{1l} = 0, \quad a_{2k}^{1l} = Y_j^* \delta_{jk}, \quad a_{3k}^{1l} = \delta_{jk} - c_{jk},$$

$$a_{1k}^{2l} = -A_j^* \delta_{jk}, \quad a_{2k}^{2l} = B_j^* Y_j^* \delta_{jk}, \quad a_{3k}^{2l} = B_j^* (\delta_{jk} - c_{jk}),$$

$$a_{1k}^{3l} = -C_j^* \delta_{jk}, \quad a_{2k}^{3l} = D_j^* Y_j^* \delta_{jk}, \quad a_{3k}^{3l} = D_j^* (\delta_{jk} - c_{jk}).$$  \hspace{1cm} (14.23)

where we have used the abbreviations

$$Y_j^* = Y_j^0 |L_j^0(P_j^0)|, \quad A_j^* = \frac{\nu_j P_j^0}{N_j^0}, \quad B_j^* = \frac{\mu_j P_j^0}{N_j^0},$$  \hspace{1cm} (14.24)
and

\[ C_j^* = \frac{\nu_j \hat{\alpha}_j}{N_j^0} Q_j^0, \quad D_j^* = \frac{\mu_j \hat{\alpha}_j}{N_j^0} Q_j^0. \]  (14.25)

Under two conditions, the $3m \times 3m$ matrix $A$ composed of the matrices $A^{il}$ can be treated as if it was a simple $3 \times 3$ matrix (i.e. as if the matrices $A^{il}$ were just real numbers):

1. all matrices $A^{il}$ are diagonal matrices, which basically requires that the input-output matrix $C$ is diagonal,
2. all constants $X_j^*$ with $X \in \{Y, A, B, C, D\}$ are independent of the economic sector $j$.

In the latter case, there exists an orthogonal matrix $R$ and an orthogonal inverse matrix $R^{-1}$ with $R^{-1}R = RR^{-1} = E = (\delta_{jk})$ so that

\[ R^{-1}CR = J, \]  (14.26)

where $J$ is the Jordan normal form or even a diagonal matrix. GILT DAS UEBERHAUPT FUER JORDAN NORMAL FORMEN ODER NUR FUER DIAGONALE MATRIZEN??

Introducing rotated vectors

\[
\begin{align*}
n'(t) &= R^{-1}n(t), \\
p'(t) &= R^{-1}p(t), \\
q'(t) &= R^{-1}q(t)
\end{align*}
\]  (14.27)

will then transform the above matrix representation into an appropriate form. Note, however, that the transformation $R$ basically defines new entities, which may be interpreted like virtual economic sectors [?] (in analogy to the “quasi-species” introduced in evolutionary biology [?]). That is, $n'_k(t) = \sum_j R^{-1}_{kj}n_j(t)$ describe variations of inventories of virtual economic sectors $k$, which are composed of fractions $R^{-1}_{kj}$ of several real economic sectors $j$. One could also say that, in such a case, there would be a better definition of mutually independent economic sectors from the point of view of mathematics.

We will now assume that either of both cases is given and that, in the second case, sectors have been defined or transformed in a way which implies that the input-output matrix $C$ corresponds to its Jordan normal form $J$. For this, the diagonal entries $J_{jj}$ correspond to the eigenvalues of the original input-output matrix $C$. Moreover, for some indices $j$ we may have entries $J_{j,j+1} = 1$, while all other matrix entries of $J = (J_{jk})$ are zero.

Let us now determine the eigenvalues $\tilde{\lambda}$ of the $3m \times 3m$ matrix $A$ composed of the matrices $A^{il}$, which determine the stability of the behavior. Making the solution ansatz

\[
\begin{align*}
n(t) &= n_0 e^{\tilde{\lambda}t}, \\
p(t) &= p_0 e^{\tilde{\lambda}t}, \\
q(t) &= q_0 e^{\tilde{\lambda}t}
\end{align*}
\]  (14.28)

and inserting this into Eq. (??) together with (??), we obtain the linear eigenvalue problem
The exponential growth, and a new system behavior will be established (but original equations (??) to (??) of the dynamic input-output model will limit the exponential growth, and a new system behavior will be established (but not a stationary one, if there is not more than one stationary solution).

In order to have a solution, the determinant of the matrix in this equation must become zero. This eventually leads to the characteristic polynomial

\[ -\tilde{\lambda}^3 \{ \tilde{\lambda}^2 + [B_j^* Y_j^* + D_j^*(1 - J_{jj})] \tilde{\lambda} + [A_j^* Y_j^* + C_j^*(1 - J_{jj})] \} = 0. \]  

(14.30)

This polynomial has the solutions \( \tilde{\lambda}_j = \lambda_0 = 0 \) \((m\) times altogether) and

\[ \tilde{\lambda}_j,\pm = -\frac{B_j^* Y_j^* + D_j^*(1 - J_{jj})}{2} \]

\[ \pm \frac{1}{2} \sqrt{[B_j^* Y_j^* + D_j^*(1 - J_{jj})]^2 - 4[A_j^* Y_j^* + C_j^*(1 - J_{jj})]} \]  

(14.31)

(which, for the plus and the minus sign, establishes another \( m \) solutions each, considering \( j \in \{ 1, \ldots, m \} \)). These solutions correspond to the \( 3m \) eigenvalues of the linear system of differential equations. More shortly, we may write

\[ \tilde{\lambda}_j,\pm \approx \frac{1}{2} \left( -A_j \pm \sqrt{(A_j)^2 - 4B_j} \right), \]  

(14.32)

with

\[ A_j = B_j^* Y_j^* + D_j^*(1 - J_{jj}) = \mu_j[C_j + \hat{\alpha}_j D_j(1 - J_{jj})], \]

\[ B_j = A_j^* Y_j^* + C_j^*(1 - J_{jj}) = \nu_j[C_j + \hat{\alpha}_j D_j(1 - J_{jj})], \]

\[ C_j = P_j^0 Y_j^* |P_j^0|^2 / N_j^0, \]

\[ D_j = Q_j^0 / N_j^0. \]  

(14.33)

Formula (14.32) becomes exact when the matrix \( C \) is diagonal or the parameters \( \mu_j, \alpha_j, \mu_j D_j, \nu_j C_j \) and \( \hat{\alpha}_j \nu_j D_j \) are sector-independent constants, otherwise the eigenvalues must be numerically determined.

Note that, depending on the eigenvalues \( J_{jj} \) of the input-output matrix \( C \), the eigenvalues \( \tilde{\lambda}_j,\pm \) may be complex. Their real value determines an exponential growth in time (if greater than zero) or decay (if less than zero). The imaginary value determines an oscillation frequency. The linear system (??) of ordinary differential equations behaves stable, if the real values of all eigenvalues are negative so that any deviation from the equilibrium state eventually decreases in time. Otherwise, if at least one of the eigenvalues \( \tilde{\lambda}_j,\pm \) has a positive value, the deviation can be assumed to grow in time. In such a case, the stationary state will be left in the course of time, and the economic equilibrium becomes meaningless. Eventually, however, the nonlinearity of the original equations (??) to (??) of the dynamic input-output model will limit the exponential growth, and a new system behavior will be established (but not a stationary one, if there is not more than one stationary solution).
14.1 “Bullwhip Effect” and “Business Cycles”

14.1.4 Eigenvalues and Dynamics of Commodity Flow Systems

Let us now study the eigenvalues of commodity flow systems. It turns out that the dynamic behavior mainly depends on the parameters $\hat{\alpha}_j$, $\nu_j / \mu_j^2$, and the eigenvalues $J_{jj}$ of the input matrix $C$ (see Fig. 14.4). Moreover, despite their mathematical similarity, Eqs. (11.29) and (11.30) have a surprisingly different impact on the dynamics of the economic system.

**Fig. 14.4.** Properties of our dynamic model of supply networks for a characteristic input matrix specified as average input matrix of macroeconomic commodity flows of several countries (top) and for a synthetic input matrix generated by random changes of input matrix entries until the number of complex eigenvalues was eventually reduced to zero (bottom). Subfigures (a), (e) illustrate the color-coded input matrices $C$, (b), (f) the corresponding network structures, when only the strongest links (commodity flows) are shown, (c), (g) the eigenvalues $J_{jj} = \text{Re}(J_{jj}) + i \text{Im}(J_{jj})$ of the respective input matrix $C$, and (d), (h) the phase diagrams indicating the stability behavior of the model equations (11.32) to (11.31) on a double-logarithmic scale as a function of the model parameters $\hat{\alpha}_j = \hat{\alpha}$ and $\nu_j / \mu_j^2 = \nu / \mu^2 = V / M^2$. The other model parameters were set to $\nu_j = C_j = D_j = P_j^0 = N_j^0 = Y_j^0 = 1$. Surprisingly, for empirical input matrices $C$, one never finds an overdamped, exponential relaxation to the stationary equilibrium state, but network-induced oscillations due to complex eigenvalues $J_{jj}$. (After Ref.[?] )

**Fast Price Adjustment**

The limiting case $\hat{\alpha}_j \to 0$ corresponds to situations where the price is adjusted much faster than the production rate. In this case, the eigenvalues $\lambda_{j,\pm}$ are given by
\[
2\tilde{\lambda}_{j,\pm} = -\mu_j C_j \pm \sqrt{(\mu_j C_j)^2 - 4\nu_j C_j},
\]
(14.34)
i.e. the network structure does not matter at all. We expect an exponential relaxation to the stationary equilibrium for \(0 < \nu_j/\mu_j^2 < C_j/4\), otherwise damped oscillations. Therefore, immediate price adjustments or similar mechanisms are an efficient way to stabilize economic and other supply systems.

In the limit of fast price adjustments, one can eliminate Eq. (14.6) by assuming \(q_j(t) \approx 0\) and \(Q_j(t) \approx Q_j^0\), so that

\[
\frac{dn_j}{dt} \approx Y_j^0 L'(P_j^0)|P_j(t) - \xi_j(t)|.
\]
(14.35)
Inserting (14.5) into the time-derivative of this equation finally results in the equations

\[
\frac{d^2 n_j}{dt^2} + \mu_j C_j \frac{dn_j}{dt} + \nu_j C_j n_j \approx -\frac{d\xi_j}{dt},
\]
(14.36)
of damped harmonic oscillators with eigenfrequencies \(\omega_j = \sqrt{\nu_j C_j}\), damping constants \(\gamma_j = \mu_j C_j / 2\), and external driving \(-d\xi_j/dt\) due to variations in the consumption rate. In Sec. ??, the solution of this equation has been discussed in detail. Here, the most important conclusion is that the economic system will eventually approach an economic equilibrium state, if price adjustments are fast enough.

**Slow Price Adjustments**

The limiting case \(\hat{\alpha}_j \gg 1\) corresponds to situations where the price adjustment is slow or the price variable is not relevant. One can then eliminate Eq. (14.5) by assuming \(p_j(t) \approx 0\) and \(P_j(t) \approx P_j^0\), so that

\[
\frac{dn_j}{dt} \approx q_j(t) - \sum_k a_{jk} q_k(t) - \xi_j(t).
\]
(14.37)
Deriving this with respect to time and inserting Eq. (14.6) delivers

\[
\frac{d^2 n_j}{dt^2} + \sum_k (\delta_{jk} - c_{jk}) \alpha_k D_k \left[ \mu_k \frac{dn_k}{dt} + \nu_k n_k(t) \right] \approx -\frac{d\xi_j}{dt}.
\]
(14.38)
If we assume sector-independent constants \(\hat{\alpha}_j \mu_j D_j = M\) and \(\hat{\alpha}_j \nu_j D_j = V\), the \(2m\) eigenvalues \(\tilde{\lambda}_{j,\pm}\) are given by

\[
2\tilde{\lambda}_{j,\pm} = -M(1 - J_{jj}) \pm \sqrt{[M(1 - J_{jj})]^2 - 4V(1 - J_{jj})}.
\]
(14.39)
For empirical input matrices \(\mathbf{C}\), one never finds an overdamped, exponential relaxation to the economic equilibrium, but network-induced oscillations (see Fig. ??). An overdamped behavior is only possible if all eigenvalues \(J_{jj}\) are real numbers.
Large Responsiveness to Changes in the Inventory

If \((A_j)^2/B_j \gg 1\), the eigenvalues become

\[\tilde{\lambda}_{j,-} \approx -A_j \quad \text{and} \quad \tilde{\lambda}_{j,+} \approx -\frac{B_j}{A_j} = -\frac{\nu_j}{\mu_j} \]  

(14.40)

(where we have used the Taylor expansion \(\sqrt{1+\epsilon} \approx 1 + \epsilon/2 + \ldots\)). This situation corresponds to a relaxation to the equilibrium state in the case of a large responsiveness \(\mu_j \gg 1\). However, this does not exclude the existence of a “bullwhip effect” (see Sec. ??). PRUEFEN!!!

An overdamped behavior in time is found if all eigenvalues \(J_{jj}\) are real numbers or if all \(\hat{\alpha}_j = 0\), otherwise one expects network-induced oscillations.

Interestingly enough, \(\mu_j \gg 1\) implies \((dP_j/dt)/(\mu_j P_j) \approx 0\), so that Eq. (11.30) reduces to

\[\frac{dN_j}{dt} \approx \frac{\nu_j}{\mu_j} [N_j^0 - N_j(t)]. \]  

(14.41)

Therefore, \(N_j(t) \approx N_j^0\) and \(dQ_j/dt \approx 0\) (i.e. \(Q_j \approx Q_j^0\)). Inserting this into Eq. (11.32) yields an implicit equation for the price \(P_j(t)\) as a function of the fluctuations \(\xi_i(t)\) in the consumption rate. It reads

\[[Y_j^0 + \xi_j(t)]L_j(P_j(t)) \approx Q_j^0 - \sum_k c_{jk}Q_k^0 = \text{const.} \]  

(14.42)

General Case

Any delay (\(\hat{\alpha}_j > 0\)) will cause complex eigenvalues \(\tilde{\lambda}_j,\pm\) and, thereby, oscillations, if the input-output matrix \(C\) has complex eigenvalues \(J_{jj} = \text{Re}(J_{jj}) + i \text{Im}(J_{jj})\). Note that, unfortunately, this is the normal case, as typical supply networks in natural and man-made systems are characterised by complex eigenvalues (see Sec. ?? and the top of Fig. 14.4).

Let us now determine under which conditions these oscillations may grow. Starting with Eq. (14.32), stability requires the real parts \(\text{Re}(\tilde{\lambda}_j)\) of all eigenvalues \(\tilde{\lambda}_j\) to be non-positive. Therefore, the stability boundary is given by \(\max_j \text{Re}(\tilde{\lambda}_j) = 0\). Writing

\[C_j + \hat{\alpha}_j D_j (1 - J_{jj}) = \hat{\theta}_j + i\hat{\beta}_j \]  

(14.43)

with \(C_j = P_j^0Y_j^0|L_j'(P_j^0)|/N_j^0\) and defining

\[\hat{\theta}_j = C_j + \hat{\alpha}_j D_j [1 - \text{Re}(J_{jj})],\]

\[\hat{\beta}_j = \pm \hat{\alpha}_j D_j \text{Im}(J_{jj}) \text{ (complex conjugate eigenvalues)},\]

\[\hat{\gamma}_j = 4\nu_j/\mu_j^2, \]  

(14.44)

we find
\[ 2\tilde{\lambda}_j/\mu_j = -\dot{\theta}_j - i\dot{\beta}_j + \sqrt{\Re_j + i\Im_j} \]  \hspace{1cm} (14.45) \{\text{realim}\}

with
\[ \Re_j = \hat{\theta}_j^2 - \hat{\beta}_j^2 - \hat{\gamma}_j\hat{\theta}_j \quad \text{and} \quad \Im_j = 2\hat{\theta}_j\hat{\beta}_j - \hat{\gamma}_j\hat{\beta}_j. \]  \hspace{1cm} (14.46)

The real part of (14.45) can be calculated, using formula (5.27), i.e.
\[ \text{Re}\left(\sqrt{\Re_j \pm i\Im_j}\right) = \sqrt{1/2 \left(\sqrt{\Re_j^2 + \Im_j^2} + \Re_j\right)}. \]  \hspace{1cm} (14.47)

The condition \( \text{Re}(2\tilde{\lambda}_j/\mu_j) = 0 \) is fulfilled by \( \hat{\gamma}_j = 0 \) and
\[ \hat{\gamma}_j = 4\hat{\theta}_j(1 + \hat{\theta}_j^2/\hat{\beta}_j^2), \]  \hspace{1cm} (14.48)

i.e. the stable regime is given by
\[ \frac{\hat{\gamma}_j}{4} = \frac{\nu_j}{\mu_j^2} = \frac{\hat{\alpha}_j\nu_j}{\mu_j^2} \leq \hat{\theta}_j \left(1 + \frac{\hat{\theta}_j^2}{\hat{\beta}_j^2}\right) \]  \hspace{1cm} (14.49)

for all \( j \), corresponding to Eq. (14.52) \{\text{new}\}.

If \( \hat{\alpha}_j > 0 \), it is, however, also possible to have real eigenvalues \( \tilde{\lambda}_j, \pm \) only. For this, the imaginary parts of all eigenvalues \( \tilde{\lambda}_j \) must vanish. This case occurs if \( \text{Im}(J_{jj}) = 0 \) (i.e. \( \hat{\beta}_j = 0 \)) and \( \Re_j \geq 0 \). This requires
\[ \frac{4\nu_j}{\mu_j^2} = \hat{\gamma}_j \leq \hat{\theta}_j - \frac{\hat{\beta}_j^2}{\hat{\theta}_j} = \hat{\theta}_j = C_j + \hat{\alpha}_j D_j(1 - J_{jj}) \]  \hspace{1cm} (14.50)

for all \( j \), corresponding to Eq. (14.53) \{\text{im}\}.

In summary, damped oscillations result if all values
\[ \frac{\nu_j}{\mu_j^2} = \frac{\hat{\alpha}_j\nu_j}{\mu_j^2} \]  \hspace{1cm} (14.51)

lie below the instability lines
\[ \{\text{new}\} \quad \frac{\nu_j}{\mu_j^2} \approx \left\{ C_j + \hat{\alpha}_j D_j[1 - \text{Re}(J_{jj})]\right\} \left(1 + \frac{\{C_j + \hat{\alpha}_j D_j[1 - \text{Re}(J_{jj})]\}^2}{[\hat{\alpha}_j D_j \text{Im}(J_{jj})]^2}\right) \]  \hspace{1cm} (14.52)

given by the condition \( \text{Re}(\tilde{\lambda}_{j, \pm}) \leq 0 \). For identical parameters \( \nu_j/\mu_j^2 = \nu/\mu^2 \) and \( \hat{\alpha}_j = \hat{\alpha} \), the minimum of these lines agrees exactly with the numerically obtained curve in Fig. 14.4. Values above this line cause small oscillations to grow over time.

In cases, where all eigenvalues \( J_{jj} \) are real (as for symmetric input-output matrices \( C \)), Eq. (14.32) predicts a stable, overdamped behavior if all values \( \nu_j/\mu_j^2 = \hat{\alpha}_j\nu_j/\mu_j^2 \) lie below the lines
\[ \{\text{im}\} \quad \frac{\nu_j}{\mu_j^2} \approx \frac{\nu}{\mu^2} \left[C_j + \hat{\alpha}_j D_j(1 - J_{jj})\right]/4 \]  \hspace{1cm} (14.53)
defined by \( \min_j (A_j^2 - 4B_j) > 0 \). For identical parameters \( \nu_j/\mu_j^2 = \nu/\mu^2 \) and \( \hat{\alpha}_j = \hat{\alpha} \), the minimum of these lines corresponds exactly to the numerically determined curve in Fig. 14.4. Above it, one observes damped oscillations around the equilibrium state, but growing oscillations are not possible. In supply systems with a slow or non-existent price adjustment mechanism (i.e. for \( \hat{\alpha}_j \gg 1 \) or \( C_j = 0 \)), Eq. (14.53) predicts an overdamped behavior for real eigenvalues \( J_{jj} \) and

\[
\frac{\hat{\nu}_j}{\hat{\mu}_j^2} < D_j(1 - J_{jj})/4
\]

for all \( j \), while Eq. (14.52) implies the stability condition

\[
\frac{\hat{\nu}_j}{\hat{\mu}_j^2} < D_j[1 - \Re(J_{jj})]|1 + |1 - \Re(J_{jj})|^2/\Im(J_{jj})^2|
\]

for all \( j \), given that some eigenvalues \( J_{jj} \) are complex. Moreover, for the case of sector-independent constants \( V = \hat{\alpha}_j \nu_j D_j \) and \( W = \hat{\alpha}_j \mu_j D_j \), the eigenvalues \( \tilde{\lambda}_j, \pm \) can be calculated as

\[
\tilde{\lambda}_j, \pm = -W(1 - J_{jj})/2 \pm \sqrt{[W(1 - J_{jj})]^2/4 - V(1 - J_{jj})}.
\]

14.1.5 Eigenvalues of Various Supply Networks

For some material flow networks, it is still possible to make predictions about the eigenvalues and the dynamics resulting from them. It is useful to distinguish the following cases:

**Symmetric Supply Networks**

If the supply matrix \( C \) is symmetric, as for most mechanical or electrical oscillator networks, all eigenvalues \( J_{ii} \) are real. Consequently, if \( \omega_i \ll \gamma_i \) (i.e. if \( V \) is small enough), the eigenvalues \( \tilde{\lambda}_i, \pm \) of \( M \) are real and negative, corresponding to an overdamped behavior. However, if \( W \) is too small, the system behavior may be characterized by damped oscillations.

**Irregular Supply Networks**

Most natural and man-made supply networks have directed links, and \( S \) is not symmetric. Therefore, some of the eigenvalues \( J_{ii} \) will normally be complex, and an overdamped behavior is untypical. The characteristic behavior is rather of oscillatory nature (although asymmetry does not always imply complex eigenvalues[?], see Fig. 14.4). For small values of \( W \), it can even happen that the real part of an eigenvalue \( \tilde{\lambda}_{i, \pm} \) becomes positive. This implies an amplification of oscillations in time (until the oscillation amplitude is limited by non-linear terms). Surprisingly, this also applies to most upper triangular matrices, i.e. when no loops in the material flows exist.
Regularly Structured Supply Networks

Another relevant case are regular supply networks. These are mostly characterized by degenerate zero eigenvalues \( J_{ii} = 0 \) and Jordan normal forms \( J \), i.e. the existence of non-vanishing upper-diagonal elements \( J_{i,i+1} \). Not only sequential supply chains, but also fully connected graphs, regular supply ladders, and regular distribution systems belong to this case [?] (see Fig. 14.5a, c, d). This is characterized by the two \( u \)-fold degenerate eigenvalues

\[
\tilde{\lambda}_\pm = -(1 + W)/2 \pm \sqrt{(1 + W)^2/4 - V},
\]

independently of the suppliers \( i \). For small enough values \( V < (1 + W)^2/4 \), the corresponding supply systems show overdamped behavior, otherwise damped oscillations.

For the purpose of illustration, the following equations display some regular input matrices \( C \) and their corresponding Jordan matrices \( J \): For a fully connected network we have

\[
C = \begin{pmatrix}
1 & 1 & 1 & 1 & \cdots & 1 \\
1 & 1 & 1 & 1 & \cdots & 1 \\
1 & 1 & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & 1 & \cdots & 1 \\
1 & 1 & 1 & 1 & \cdots & 1 \\
1 & 1 & 1 & 1 & \cdots & 1
\end{pmatrix}
\quad \text{and} \quad
J = \begin{pmatrix}
0 & 0 & 0 & 0 \cdots & 0 \\
0 & 1 & 0 & 0 \cdots & 0 \\
0 & 0 & 0 & 0 \cdots & 0 \\
0 & 0 & 0 & 0 \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
0 & 0 & 0 & 0 & \cdots & 0
\end{pmatrix}.
\]
The Jordan normal matrix $J$ of a sequential supply chain corresponds to the input matrix $C$ itself, i.e. $J = C$. This, however, is quite exceptional. For the supply ladder displayed in Fig. 14.5c we have

$$C = \begin{pmatrix}
0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}$$

and

$$J = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}, \quad (14.59)$$

where the number of ones corresponds to the number of levels of the supply ladder. For the hierarchical distribution network shown in Fig. 14.5d, but with 3 levels only, we have

$$C = \begin{pmatrix}
0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\
0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\
0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
0 & 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\end{pmatrix}$$

and

$$J = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}, \quad (14.60)$$

Note that the Jordan normal forms of different input matrices $C$ may be identical, but the transformation matrix $T$ and the driving term $h(t)$ would then be different.

**Regularly Structured Supply Networks with Perturbations**

Randomized regular supply networks belong to the class of irregular supply networks, but they can be viewed as slightly perturbed regular supply networks. For this reason, there exist approximate analytical results for their eigenvalues. Even very small perturbations of the regular matrices $S$ discussed in the previous paragraph can change the eigenvalue spectrum qualitatively. Instead of the two multiply degenerate eigenvalues $\tilde{\lambda}_\pm$ of Eq. (??), we find a scattering of eigenvalues around these values. The question is: why?

In order to assess the behavior of randomized regular supply networks, we apply Geršgorin’s theorem on the location of eigenvalues.[?] According to this, the $n$ complex eigenvalues $\tilde{\lambda}_k \in \mathbb{C}$ of some $n \times n$-matrix $N$ are located in the union of $n$ disks:
\[ \tilde{\lambda}_k \in \bigcup_i \left\{ z \in \mathbb{C} : |z - N_{ii}| \leq \sum_{j \neq i} |N_{ij}| \right\} \quad (14.61) \]

Furthermore, if a union of \( l \) of these \( n \) discs form a connected region that is disjoint from all the remaining \( n - l \) discs, then there are precisely \( l \) eigenvalues of \( N \) in this region.

Let us now apply this theorem to the perturbed matrix

\[ C_\eta = C + \eta P. \quad (14.62) \]

For small enough values of \( \eta \), the corresponding eigenvalues \( J_{ii} \) should be located within discs of radius

\[ R_i(\eta) = \eta \sum_j |P_{ij}^{(\eta)}| \quad (14.63) \]

around the (possibly degenerated) eigenvalues \( J_{ii} \) of the original matrix \( C \) (see Fig. 14.6). This radius grows monotonously, but not necessarily linearly in the parameter \( \eta \) with \( 0 < \eta \leq 1 \), which allows to control the size of the perturbation. Moreover, \( P^{(\eta)} = R_\eta^{-1} P R_\eta \), where \( R_\eta \) is the orthogonal matrix which transforms \( C_\eta \) to a diagonal matrix \( D^{(\eta)} \), i.e. \( R_\eta^{-1} C_\eta R_\eta = D^{(\eta)} \). (This assumes a perturbed matrix \( C_\eta \) with no degenerate eigenvalues.) Similar discs as for the eigenvalues of \( C_\eta \) can be determined for the associated eigenvalues \( \tilde{\lambda}_{i,\pm}^{(\eta)} \) of the perturbed \( (2u \times 2u) \)-matrix \( M_\eta \) belonging to the perturbed \( u \times u \)-matrix \( C_\eta \), see Eq. (??) and Fig. 14.6.

![Fig. 14.6.](image)

Let us now discuss the example of a structural perturbation of a sequential supply chain (with \( \eta = 0 \)) towards a supply circle (with \( \eta = 1 \)), see Fig. 14.5b. For this, we set
While the normal form for $\eta = 0$ is given by a Jordan matrix $J$ which agrees with $C$, for any $\eta > 0$ we find the diagonal matrix

$$J = \begin{pmatrix} J_{11} & 0 & \cdots & 0 \\ 0 & J_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_{nn} \end{pmatrix},$$

(14.65) \{circle\}

where the diagonal elements $J_{ii}$ are complex and equally distributed on a circle of radius $\sqrt{\eta}$ around the origin of the complex plane. Therefore, even an arbitrarily small perturbation can change the eigenvalues qualitatively and remove the degeneration of the eigenvalues.

Connection between Cycles and Eigenvalues

Inverse of a 2x2 Matrix

Determine the inverse matrix $M^{-1}$ of a 2x2 matrix $M$, i.e. find the matrix $M^{-1}$ that fulfills $M^{-1}M = 1$ and show that $MM^{-1} = 1$ holds as well.

Let us assume

$$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}. \quad (14.66)$$

The matrix product $P = LM$ is defined by

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \quad (14.67)$$

with $P_{jk} = \sum_j L_{ij}M_{jk}$. Hence, if we want $P$ to be the unity matrix $1$, we should have

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} L_{11}M_{11} + L_{12}M_{21} & L_{11}M_{12} + L_{12}M_{22} \\ L_{21}M_{11} + L_{22}M_{21} & L_{21}M_{12} + L_{22}M_{22} \end{pmatrix}. \quad (14.68)$$

This implies the relationships

$$L_{11}M_{11} + L_{12}M_{21} = 1, \quad (14.69)$$
$$L_{11}M_{12} + L_{12}M_{22} = 0, \quad (14.70)$$
$$L_{21}M_{11} + L_{22}M_{21} = 0, \quad (14.71)$$
$$L_{21}M_{12} + L_{22}M_{22} = 1. \quad (14.72)$$
From Eqs. (14.70) and (14.71) it follows that

\[ L_{11} = -L_{12}M_{22}/M_{12}, \quad (14.73) \]
\[ L_{21} = -L_{22}M_{21}/M_{11}. \quad (14.74) \]

Inserting Eq. (14.73) into Eq. (14.69) and Eq. (14.74) into Eq. (14.72), and using the abbreviation

\[ \det(M) = M_{11}M_{22} - M_{21}M_{12} \quad (14.75) \]

gives

\[ L_{12} = -M_{12}/\det(M), \quad (14.76) \]
\[ L_{22} = M_{11}/\det(M). \quad (14.77) \]

Inserting Eq. (14.76) into Eq. (14.73) and Eq. (14.77) into Eq. (14.74) finally results in

\[ L_{11} = M_{22}/\det(M), \quad (14.78) \]
\[ L_{21} = -M_{21}/\det(M). \quad (14.79) \]

Hence, altogether we have

\[ M^{-1} = L = \frac{1}{\det(M)} \begin{pmatrix} M_{22} & -M_{12} \\ -M_{21} & M_{11} \end{pmatrix}. \quad (14.80) \]

Moreover, as

\[ \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} M_{22} & -M_{12} \\ -M_{21} & M_{11} \end{pmatrix} = \begin{pmatrix} M_{12}M_{21} - M_{11}M_{22} - M_{11}M_{12} + M_{12}M_{11} \\ M_{21}M_{22} - M_{22}M_{21} - M_{12}M_{21} + M_{22}M_{11} \end{pmatrix}, \]
\[ = \det(M) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (14.81) \]

we also have \( MM^{-1} = I \), as was to be shown.

### Roots of a Quadratic Equation

Derive formulas for the solutions (the so-called "roots") of the quadratic equation

\[ Ax^2 + Bx + C = 0. \quad (14.82) \]

The problem is solved, if we can find two values \( x_- = \overline{x} - \Delta x \) and \( x_+ = \overline{x} + \Delta x \) with

\[ Ax^2 + Bx + C = A(x - x_-)(x - x_+), \quad (14.83) \]

because then, \( x_- \) and \( x_+ \) are the roots of the quadratic equation \( Ax^2 + Bx + C = 0 \).

In order to determine these values, we consider

\[ A(x - x_-)(x - x_+) = A(x^2 - xx_- - xx_+ + x_-x_+). \quad (14.84) \]

As a consequence, we have the conditions

\[ B = -A(x_- + x_+) = -2Ax, \quad (14.85) \]
\[ C = Ax_-x_+ = A(\overline{x} - \Delta x)(\overline{x} + \Delta x) = A[\overline{x}^2 - (\Delta x)^2]. \quad (14.86) \]
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From Eq. (14.85) we obtain \( x = -\frac{B}{2A} \), and inserting this into Eq. (14.86) gives

\[
(\Delta x)^2 = \frac{1}{4A^2} (B - 4AC) \quad \text{or} \quad \Delta x = \pm \frac{1}{2A} \sqrt{B^2 - 4AC}.
\] (14.87)

 Altogether, the two solutions of the quadratic equation (14.82) have been identified as

\[
x_+ = -\frac{B - \sqrt{B^2 - 4AC}}{2A} \quad \text{and} \quad x_- = -\frac{B + \sqrt{B^2 - 4AC}}{2A}.
\] (14.88)

\textbf{Mitternachtsformel}

\textbf{HAT DIESE FORMEL IM ENGLISCHEN EINEN NAMEN?}

Moments of a Normal Gaussian Distribution

Determine the \( n \)-th moment \( \langle x^n \rangle := \int_0^\infty dx \, x^n e^{-x^2/(2\Theta)} \) (14.89)

We apply partial integration (see Exercise ...) with \( u(x) = x^n \) and \( v(x) = e^{-x^2/(2\Theta)} \). The chain rule of calculus gives

\[
v'(x) = -\left(\frac{x}{\Theta}\right) e^{-x^2/(2\Theta)}.
\]

Considering this together with \( u(0) = 0^{n+1}/(n+1) = 0 \) and \( v(\infty) = e^{-\infty^2/(2\Theta)} = 0 \), partial differentiation yields the relationship

\[
\int_0^\infty dx \, e^{-x^2/(2\Theta)} = \frac{1}{\Theta} \int_0^\infty dx \, x^n+2 e^{-x^2/(2\Theta)}.
\] (14.90)

Therefore, \( (x^n+2) = (n+1)\Theta(x^n) \). Applying this relationship recursively (i.e. inserting it again and again), we find

\[
\langle x^{2k} \rangle = (2k-1) \cdot \ldots \cdot 3 \cdot 1 \Theta^k \langle x^0 \rangle
\] (14.91)

for \( n = 2k \) and

\[
\langle x^{2k+1} \rangle = k! (2\Theta)^k \langle x^1 \rangle
\] (14.92)

for \( n = 2k+1 \), where \( k! = k \cdot (k-1) \cdot \ldots \cdot 2 \cdot 1 \) and \( 0! = 1 \). \( \langle x \rangle \) can be determined by transformation of variables: For \( y = x^2 \) we get \( dy = 2x \, dx \) or \( dx = dy/(2x) \), which implies

\[
\langle x^1 \rangle = \int_0^\infty dx \, x e^{-x^2/(2\Theta)} = \frac{1}{2} \int_0^\infty dy \, e^{-y/(2\Theta)} = \frac{1}{2} \left[ -2\Theta e^{-y^2/(2\Theta)} \right]_0^\infty = -0 + \Theta.
\] (14.93)

Finally, we observe that

\[
\int_{-\infty}^\infty dx \, e^{-x^2/(2\Theta)} \cdot \int_{-\infty}^\infty dy \, e^{-y^2/(2\Theta)} = (2\langle x^0 \rangle)^2.
\] (14.94)

The product of the integrals over \( x \) and over \( y \) can be rewritten as
where we have used polar coordinates \( r \) and \( \varphi \) with \( x = r \cos \varphi \) and \( y = r \sin \varphi \), which implies \( dx \, dy = r \, d\varphi \, dr \) and \( x^2 + y^2 = r^2 (\cos^2 \varphi + \sin^2 \varphi) = r^2 \), see ...............

In conclusion, we have

\[
\langle x^0 \rangle = \pi \Theta \frac{\Theta}{2}. \tag{14.96}
\]

With this, we can determine the even moments of the Gaussian distribution \( e^{-x^2/(2\Theta)}/\sqrt{2\pi \Theta} \) as

\[
\int_{-\infty}^{\infty} dx \, x^{2k} \frac{e^{-x^2/(2\Theta)}}{\sqrt{2\pi \Theta}} = 1 \cdot 3 \cdot \ldots \cdot (2k-1)\Theta^k \tag{14.97}
\]

(which is 1 for \( k = 0 \)), while all uneven moments vanish, as \( f(x) = x^{2k+1} e^{-x^2/(2\Theta)} \) is asymmetric with respect to the origin at \( x = 0 \), i.e. \( f(-x) = -f(x) \). Note that we get the moments of the normal distribution simply by setting the variance \( \Theta \) (i.e. the second moment) to unity (\( \Theta = 1 \)).

### Solution of a System of Ordinary Differential Equations with Constant Coefficients

Let us assume an inhomogeneous system of ordinary differential equations of the form

\[
\frac{dX}{dt} = AX(t) + B(t) \tag{14.98}
\]

with a time-independent matrix \( A = (A_{ij}) \) (i.e. constant coefficients \( A_{ij} \)). Moreover, let \( \Phi(t) \) be a fundamental system, i.e.

\[
\frac{d\Phi}{dt} = A \Phi(t). \tag{14.99}
\]

Show that \( X(t) = X^{\text{hom}}(t) = \Phi(t)X_0 \) solves the homogeneous system of differential equations \( dX/dt = AX(t) \) for any choice of \( X_0 \) and that \( X(t) = X^{\text{inh}}(t) \) with

\[
X^{\text{inh}}(t) = \Phi(t) \int_0^t dt' \Phi^{-1}(t') B(t') \tag{14.100}
\]

is a particular solution of the inhomogeneous system of differential equations (14.98). Furthermore, demonstrate that

\[
X(t) = X^{\text{hom}}(t) + X^{\text{inh}}(t) = \Phi(t)X_0 + \Phi(t) \int_0^t dt' \Phi^{-1}(t') B(t') \tag{14.101}
\]

is the general solution of Eq. (14.98). How must \( X_0 \) be chosen in order to be consistent with the initial condition \( X(0) \)?

From \( X^{\text{hom}}(t) = \Phi(t)X_0 \) and with Eq. (14.99) we obtain
so that $X(t) = X^\text{hom}(t)$ satisfies the homogeneous system of differential equations $dX/dt = A X(t)$ for any choice of $X_0$. Furthermore, for $X^\text{inh}(t)$ as defined in Eq. (14.100) we get
\[
\frac{dX^\text{inh}}{dt} = \frac{d\Phi}{dt} \int_0^t dt' \Phi^{-1}(t') B(t') + \Phi(t) \Phi^{-1}(t) B(t) = A \Phi(t) \int_0^t dt' \Phi^{-1}(t') B(t') + 1 B(t) = A X^\text{inh}(t) + B(t),
\]
where we have applied Eq. (14.99) and the fundamental theorem of calculus, i.e.
\[
\frac{d}{dt} \int_{t_0}^t dt' f(t') = f(t).
\]
Finally, for $X(t) = X^\text{hom}(t) + X^\text{inh}(t)$ we find
\[
\frac{dX}{dt} = \frac{dX^\text{hom}}{dt} + \frac{dX^\text{inh}}{dt} = A X^\text{hom}(t) + A X^\text{inh}(t) + B(t) = A X(t) + B(t)
\]
for any choice of $X_0$, so Eq. (14.101) constitutes a general solution of the inhomogeneous system of ordinary differential equations (14.98). From the definition of Eq. (14.100) we can see $X^\text{inh}(0) = 0$. Therefore, $X(0) = X^\text{hom}(0) = \Phi(0) X_0$. This shows that we must choose
\[
X_0 = \Phi^{-1}(0) X(0)
\]
in order to satisfy the initial condition $X(0)$.

**Fundamental System of the Damped Harmonic Oscillator**

Assume
\[
\frac{dX}{dt} = \frac{d}{dt} \begin{pmatrix} X_j(t) \\ U_j(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ - (\omega_0)^2 & -2\gamma \end{pmatrix} \begin{pmatrix} X_j(t) \\ U_j(t) \end{pmatrix} = \Delta X(t)
\]
and show that
\[
\Phi(t) = \begin{pmatrix} e^{\tilde{\lambda}_+ t} & e^{\tilde{\lambda}_- t} \\ \tilde{\lambda}_+ e^{\tilde{\lambda}_+ t} & \tilde{\lambda}_- e^{\tilde{\lambda}_- t} \end{pmatrix}
\]
with
\[
\tilde{\lambda}_\pm = -\gamma \pm \sqrt{\gamma^2 - (\omega_0)^2}
\]
is a fundamental system.

We have to show that
\[
\frac{d\Phi}{dt} = \begin{pmatrix} \tilde{\lambda}_+ e^{\tilde{\lambda}_+ t} & \tilde{\lambda}_- e^{\tilde{\lambda}_- t} \\ (\tilde{\lambda}_+)^2 e^{\tilde{\lambda}_+ t} & (\tilde{\lambda}_-)^2 e^{\tilde{\lambda}_- t} \end{pmatrix}
\]
where
\[
\frac{dX^\text{hom}}{dt} = \frac{d\Phi}{dt} X_0 = A \Phi(t) X_0 = A X^\text{hom}(t),
\]
agrees with
\[
\Phi(t) = \begin{pmatrix}
0 & 1 \\
-(\omega_0)^2 - 2\gamma & \lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}
\end{pmatrix}
\begin{pmatrix}
e^{\lambda_+ t} & e^{\lambda_- t}
\end{pmatrix}
\begin{pmatrix}
\lambda_+ e^{\lambda_+ t} & \lambda_- e^{\lambda_- t}
\end{pmatrix}
\]
\begin{equation}
(14.111)
\end{equation}

This requires that the following two equations are fulfilled:
\begin{align}
(\lambda_+)^2 + 2\gamma \lambda_+ + (\omega_0)^2 &= 0, \\
(\lambda_-)^2 + 2\gamma \lambda_- + (\omega_0)^2 &= 0.
\end{align}
\begin{equation}
(14.112) \quad \text{and} \quad (14.113)
\end{equation}

These equations are indeed satisfied due to Eq. (14.109).

**Condition for Accident-Free Driving and Non-Negative Velocities in the Optimal Velocity Model**

Assume the optimal velocity model (2.25) together with the equation (2.26) for the change in the vehicle distance and the optimal velocity function
\begin{equation}
v_o(d_i) = \max \left[0, \min \left( v^0, \frac{d - d_0}{T} \right) \right]
\end{equation}
\begin{equation}
(14.114)
\end{equation}
with positive parameters \(v^0, d_0,\) and \(T\). Provide an interpretation of this optimal velocity function. Derive a condition, for which the approach of a slower vehicle is always non-oscillatory and accident-free. Show that the vehicle velocity can become negative, if the linear instability condition \((??)\) is fulfilled. How can a more realistic car-following model be built?

The function (14.114) limits the optimal velocity to the range between 0 and \(v^0\) and assumes a safe distance of \(d_i = d_0 + T v_i\), which increases with the vehicle speed. \(T\) should be the safe time gap, and \(d_0\) should be somewhat greater than the vehicle length \(l_i\).

The worst-case deceleration maneuver occurs if \(v_{i-1}(t) = 0\) and \(v_i(0) = v^0\). We will assume that, at time \(t = 0\), vehicle \(i\) has the safe distance \(d_i(0) = d_0 + T v^0\). Then, we have to solve the following equations:
\begin{equation}
\frac{dv_i}{dt} = \frac{1}{\tau} \left[ v_o(d_i(t)) - v_i(t) \right] = \frac{1}{\tau} \left[ \frac{d_i(t) - d_0}{T} - v_i(t) \right],
\end{equation}
\begin{equation}
(14.115)
\end{equation}
\begin{equation}
\frac{dd_i}{dt} = v_{i-1}(t) - v_i(t) = -v_i(t).
\end{equation}
\begin{equation}
(14.116)
\end{equation}

Deriving Eq. (14.116) with respect to time and inserting Eq. (14.115) results in
\begin{equation}
\frac{d^2d_i}{dt^2} = -\frac{dv_i}{dt} = -\frac{1}{\tau} \left[ \frac{d_i(t) - d_0}{T} - v_i(t) \right]
\end{equation}
\begin{equation}
(14.117)
\end{equation}
or
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\[
\frac{d^2d_i}{dt^2} + \frac{1}{\tau} \frac{dd_i}{dt} + \frac{d_i(t)}{\tau} = \frac{d_0}{\tau T}.
\]  

(14.118)

When defining \(X_i(t) = d_i(t) - d_0\), \(\gamma = 1/\tau\), and \((\omega_0)^2 = 1/(\tau T)\), this exactly agrees with Eq. (6.9) for a damped oscillator. Therefore, the eigenvalues are given by

\[
\lambda_{\pm} = -\gamma \pm \sqrt{\gamma^2 - (\omega_0)^2} = -\frac{1}{2\tau} \pm \sqrt{\frac{1}{4\tau^2} - \frac{1}{\tau T}},
\]  

(14.119)

and the general solution is

\[
X_i(t) = X_i^{\text{hom}}(t) = X_i^+ e^{\lambda t} + X_i^- e^{-\lambda t},
\]  

(14.120)

see Eq. (6.16). If \(\gamma^2 \geq (\omega_0)^2\), which requires \(T \geq 4\tau\), the solution is a superposition of two exponential decays. Therefore, no oscillations will occur, and we have \(X_i(t) = d_i(t) - d_0 \geq 0\), which means that no accidents will happen. Moreover, \(v_i(t) = -dd_i/dt = -dX_i/dt \geq 0\), which implies non-negative velocities.

Considering \(dv_i/dx = 1/T\) for the specification (14.114) of the optimal velocity model, however, the condition (5.57) for non-linear instability reads

\[
\frac{1}{T} > \frac{1}{2\tau} \quad \text{or} \quad T < 2\tau.
\]  

(14.121)

As a consequence, traffic jams cannot emerge under the above discussed condition \(T \geq 4\tau\). In the case \(T < 4\tau\), the general solution is given by

\[
X_i(t) = X_i^{\text{hom}}(t) = [X_i^+ \cos(\omega t) + X_i^- \sin(\omega t)] e^{-t/(2\tau)}
\]  

(14.122)

with

\[
\omega = \sqrt{(\omega_0)^2 - \gamma^2} = \sqrt{\frac{1}{\tau T} - \frac{1}{4\tau^2}},
\]  

(14.123)

see Eqs. (6.17) and (6.18). Assuming the initial conditions \(X_i(0) = v^{0T}\) and \(dX_i(0)/dt = -v_i(0) = -v^0\), we can determine the parameters \(X_i^1\) and \(X_i^2\) as

\[
X_i^1 = v^{0T} \quad \text{and} \quad X_i^2 = v^0 \left( \frac{T}{2\tau} - 1 \right).
\]  

(14.124)

The time-dependence of the velocity \(v_i(t)\) can be eventually determined as

\[
v_i(t) = -\frac{dX_i}{dt} = v^0 \left[ \cos(\omega t) + \frac{1}{2\tau \omega} \sin(\omega t) \right] e^{-t/(2\tau)}.
\]  

(14.125)

It can be seen that the speed \(v_i(t)\) oscillates in time and can become negative.

In summary, the above specification of the optimal velocity model behaves non-oscillatory and accident-free, or it can describe emergent traffic, but then vehicle velocities can become negative. Besides, there is a parameter range \(2\tau < T < 4\tau\), in which we expect oscillatory behavior, but no linear instability. That means the above model cannot describe the empirical observations well. However, the problem can be solved by another speed-distance relationship with steeper slopes at medium distances. The specification (???), for example, imply linear instability without producing negative velocities or accidents, if simulations are performed for a circular road with almost homogeneous initial conditions, and if \(\tau\) and \(d_0\) are properly chosen. Another possibility is to make the optimal velocity dependent on the relative
velocity \( v_i(t) - v_{i-1}(t) \). In this way, it can be described that an acceleration maneuver is started earlier, if the relative velocity is high. We should also note that a realistic model does not to be accident-free in the worst-case scenario. In reality, such scenarios will often lead to accidents. Therefore, drivers will avoid the build-up of such worst-case scenarios. Rather than assuming a relative velocity of \( v^0 \) as in the worst-case scenario considered above, more realistic safety analyses would assume a relative velocity of

\[
|v_i(t) - v_{i-1}(t)| \approx f_s \frac{d v_o}{d t} d_i(t),
\]

where \( f_s \geq 1 \) is a safety parameter. DOES THIS MAKE SENSE?

**Application of the Rule of de L’Hospital??**

Derive from Eq. (6.25) the limiting cases \( \lambda \to \lambda_- \) and \( \lambda \to \lambda_+ \). As the denominator of Eq. (6.25) is vanishing in these cases, apply the rule of de L’Hospital???:

If \( f(0) = 0 \) and \( g(0) = 0 \), then

\[
\lim_{\Delta \to 0} \frac{f(\Delta)}{g(\Delta)} = \lim_{\Delta \to 0} \frac{\frac{df(\Delta)}{d\Delta}}{\frac{dg(\Delta)}{d\Delta}}.
\]

To study the limiting case \( \lambda \to \lambda_- \), we define \( \lambda = \lambda_- + \Delta \) and rewrite Eq. (??):

\[
X_j^{inh}(t) = X_{j-1}^0(\omega_0)^2 \frac{(\lambda_- - \lambda_+)(\lambda_- + \Delta - \lambda_+ \Delta)}{(\lambda_- - \lambda_+)^2}.
\]

Applying the rule of de L’Hospital, in the limit \( \Delta \to 0 \) we obtain

\[
X_j^{inh}(t) = X_{j-1}^0(\omega_0)^2 \frac{(\lambda_- - \lambda_+)^2}{(\lambda_- - \lambda_+)^2} \frac{e^{\lambda_- t} e^{\lambda_+ t} - e^{\lambda_- t}}{e^0 + e^{\lambda_+ t} - e^{\lambda_- t}}
\]

\[
= -X_{j-1}^0(\omega_0)^2 \frac{1}{(\lambda_- - \lambda_+)^2} \frac{e^{\lambda_- t} e^{\lambda_+ t} - e^{\lambda_- t}}{e^0 + e^{\lambda_+ t} - e^{\lambda_- t}}.
\]

Analogously, in the limit \( \lambda \to \lambda_+ \) we get

\[
X_j^{inh}(t) = -X_{j-1}^0(\omega_0)^2 \frac{1}{(\lambda_- - \lambda_+)^2} \frac{e^{\lambda_- t} e^{\lambda_+ t} - e^{\lambda_- t}}{e^0 + e^{\lambda_+ t} - e^{\lambda_- t}}.
\]

For \( \lambda_{\pm} = -\gamma \pm i \omega \), we have \( (\lambda_+ - \lambda_-) = 2i\omega \) and \( (\lambda_+ - \lambda_-)^2 = -4\omega^2 \), which implies

\[
X_j^{inh}(t) = X_{j-1}^0(\omega_0)^2 \frac{1 - 2i\omega t}{4\omega^2} e^{\lambda_+ t} - e^{\lambda_- t}.
\]

Note that the prefactor \( t \) of one of the exponential terms is typical, if the same eigenvalue occurs twice (i.e. in case of a degenerate eigenvalue of multiplicity 2).
Problem Solutions

Exercises of Chapter 2

Exercise 2.1

For increasing values of $A$, the repulsive interaction forces between two pedestrians become proportionally larger, independently of the distance. The parameter $B$ determines the distance-dependence and has the interpretation of the interaction range, i.e. the distance at which the force has a value of $Ae^{-1}$.

Exercise 2.2

Let us start with the case in which $v_i(0)$ points into the desired direction $e_0^i$ of motion. Then, we have $v_i(t) = v_i(t)e_0^i$ with

$$\frac{dv_i(t)}{dt} = \frac{v_0^i - v_i(t)}{\tau}.$$  \hfill (15.1) \{inhom\}

We first solve the homogeneous linear differential equation

$$\frac{dv_i(t)}{dt} = \frac{v_i(t)}{\tau}.$$  \hfill (15.2)

Dividing this by $v_i(t) > 0$, we can separate the variables:

$$\frac{dv_i}{v_i(t)} = \frac{d\ln[v_i(t)]}{dt} = -\frac{1}{\tau}.$$  \hfill (15.3)

Integrating this over time results in

$$\ln v_i(t) - \ln A = -\frac{t}{\tau} \quad \text{or} \quad v_i(t) = Ae^{-t/\tau} = v_i^{\text{hom}}(t),$$  \hfill (15.4)

where $A$ is an integration constant. After having identified this general solution of the homogeneous differential equation, we determine a special solution of the inhomogeneous linear differential equation (15.1) via the well-known formula [?].
\[ v_i(t) = B v_{ih}^{\text{hom}}(t) \int_0^t dt' \frac{1}{v_{ih}^{\text{hom}}(t')} \frac{v_i^0}{\tau} \]
\[ = B A e^{-t/\tau} \int_0^t dt' \frac{1}{A} e^{t'/\tau} \frac{v_i^0}{\tau} \]
\[ = \frac{B v_i^0}{\tau} e^{-t/\tau} \left[ e^{t/\tau} - 1 \right] \]
\[ = \frac{B v_i^0}{\tau} \left[ 1 - e^{-t/\tau} \right] = v_{ih}^i(t). \quad (15.5) \]

The general solution of Eq. (15.1) is given as
\[ v_i(t) = v_{ih}^{\text{hom}}(t) + v_i^{\text{inh}}(t) \]. The parameters \( A \) and \( B \) must be determined in a way that is consistent with the initial conditions \( v_i(0) \) and \( \frac{dv_i(0)}{dt} = \left[ v_i^0 - v_i(0) \right]/\tau \). This finally gives the special solution
\[ v_i(t) = v_i^0 + \left[ v_i(0) - v_i^0 \right] e^{-t/\tau}. \quad (15.6) \]

As can be easily checked by differentiation, the special solution of fully multidimensional case of Eq. (??) is
\[ \mathbf{v}_i(t) = \mathbf{v}_i^0 e_{i0}^0 + \left[ \mathbf{v}_i(0) - \mathbf{v}_i^0 e_{i0}^0 \right] e^{-t/\tau}. \quad (15.7) \]

In the limit \( t \to \infty \), the velocity converges to the desired velocity \( v_i^0 e_{i0}^0 \).

**Exercise 2.3**

Considering the insights gained in Exercise 2.2, the function \( v_e \left( d_i(t), v_i(t), \Delta v_i(t) \right) \) is something like a dynamic equilibrium velocity, to which the actual velocity \( v_i(t) \) tries to adapt. In the stationary case and for identical vehicles, we have \( dd_i(t)/dt = 0 \) and \( dv_i(t)/dt = 0 \). As we have assumed \( N \) vehicles on a circular road of length \( L \), we furthermore have \( d_i = L/N \). Finally, we can take into account \( \Delta v_i = 0 \) to obtain
\[ \frac{dv_i(t)}{dt} = 0 = \frac{1}{\tau} \left[ v_e(L/N, v_i, 0) - v_i \right]. \quad (15.8) \]

This is an implicit equation which allows to determine the stationary speed \( v_i \) as a function of the average vehicle density \( \rho = N/L \) on the circular road. Obviously, however, the dynamic speed \( v_i(t) \) as a function of the local density \( 1/d_i(t) \) is not unique.

**Exercise 2.4**

Using the total time derivative, which describes the density change from the perspective of a driver moving at the speed \( V(t) \), the continuity equation reads
\[ \left( \frac{\partial}{\partial t} + V(x,t) \frac{\partial V(x,t)}{\partial x} \right) \rho(x,t) = \frac{d\rho(x,t)}{dt} = -\rho \frac{\partial V(x,t)}{\partial x}. \quad (15.9) \]

Now, assume that \( \rho(x,t) > 0 \) at time \( t = 0 \) and that the density \( \rho(x,t) \) would cross the value zero from above for the first time at \( x = x_\ast \), when \( t = t_\ast > 0 \). This, however, implies
\[
\frac{d\rho(x_*, t_*)}{dt} = -\rho(x_*, t_*) \frac{\partial V(x_*, t_*)}{\partial x} = 0 ,
\]  
so that the density cannot further decrease. As a consequence, the density cannot become negative.

Exercise 2.5

Let us solve both questions at the same time by assuming \( \rho(x, t) = I(x, t)\rho'(x, t) \). Inserting this into the continuity equation (??) and performing the differentiation, applying the product rule of Calculus, we get

\[
\frac{\partial}{\partial t} [I(x, t)\rho'(x, t)] = \frac{\partial I(x, t)}{\partial t} \rho'(x, t) + I(x, t) \frac{\partial \rho'(x, t)}{\partial t}
\]

\[
= - \frac{\partial}{\partial x} [I(x, t)\rho'(x, t)V(x, t)]
\]

\[
= -\rho'(x, t)V(x, t) \frac{\partial I(x, t)}{\partial x} - I(x, t) \frac{\partial}{\partial x} [\rho'(x, t)V(x, t)]
\]  
(15.11)

Rearranging the different terms, we find

\[
\frac{\partial \rho'(x, t)}{\partial t} + \frac{\partial}{\partial x} [\rho'(x, t)V(x, t)] = - \rho'(x, t)V(x, t) \frac{\partial I(x, t)}{\partial x} - I(x, t) \frac{\partial}{\partial x} [\rho'(x, t)V(x, t)]
\]  
(15.12)  
\{with change\}

The first term of this equation exactly remind of the continuity equation for the density \( \rho(x, t) \) over the whole cross section at \( x \). The first term on the right-hand side of the equality sign describes an increase of the density, whenever the width decreases (\( \partial I(x, t)/\partial x < 0 \)). This happens, for example, when the number of freeway lanes is reduced and all vehicles squeeze into the remaining lanes. In contrast, the density per lane \( \rho'(x, t) \) goes down, if the width of the road increases (\( \partial I(x, t)/\partial x > 0 \)). A change of the width of a road with time may occur in pedestrian flows, if some pedestrians stop or take a rest along the roadside. As expected, such effects increase the specific density \( \rho'(x, t) \) of moving pedestrians, where the width is reduced (\( \partial I(x, t)/\partial t > 0 \)).

Exercise 2.6

Let us start with the application of the production rule to \( u(x)v(x) \), which leads to

\[
\frac{d}{dx} [u(x)v(x)] = \frac{du(x)}{dx} v(x) + u(x) \frac{dv(x)}{dx} .
\]  
(15.13)

Integration over \( x \) and application of the main theorem of differential and integral calculus???????,

\[
\int_0^L dx \frac{dF(x)}{dx} = F(L) - F(0) ,
\]  
(15.14)

eventually gives

\[
\int_0^L dx u(x) \frac{dv(x)}{dx} = [u(L)v(L) - u(0)v(0)] - \int_0^L dx u(x) \frac{dv(x)}{dx} .
\]  
(15.15)
This is known as the formula of partial integration.

The derivation of the formula for partial integration in \( n \) dimensions again starts with the application of the product rule, which results in

\[
\nabla_x \cdot [u(x)v(x)] = [\nabla_x u(x)] \cdot v(x) + u(x)[\nabla_x \cdot v(x)].
\] (15.16)

We integrate this, using Gauss' integral theorem

\[
\int_V d^n x \nabla_x \cdot F(x) = \oint_{\partial V} dA \cdot F(x)
\] (15.17)

and rearrange terms to obtain

\[
\int_V d^n x \left[ \nabla_x u(x) \right] \cdot v(x) = \int_V d^n x \left[ u(x)v(x) \right] - \int_V d^n x u(x) \left[ \nabla_x \cdot v(x) \right]
\]

\[
= \oint_{\partial V} dA \cdot \left[ u(x)v(x) \right] - \int_V d^n x u(x) \left[ \nabla_x \cdot v(x) \right].
\] (15.18)

**Exercise 2.7**

To derive the \( n \)-dimensional continuity equation, let us study an \( n \)-dimensional hyper-cube with the point \( x \) in its center and edges of length \( \Delta x \). Its corners shall be given by the coordinates \((x_1 \pm \Delta x/2, x_2, \ldots, x_n \pm \Delta x/2)\). For \( n = 3 \), the hypercube corresponds to a normal cube of volume \( \Delta x^3 \), for \( n = 2 \) to a square of area \( \Delta x^2 \), and for \( n = 1 \) to a line of length \( \Delta x \). Assuming a conservation of the number \( N(x, t) \) of elements in each infinitesimal (hyper-)cube of volume \( \Delta x^n \), the change

\[
\frac{dN(x, t)}{dt} = \int_{x_1 - \Delta x/2}^{x_1 + \Delta x/2} \cdots \int_{x_n - \Delta x/2}^{x_n + \Delta x/2} \frac{\partial \rho(x, t)}{\partial t} \approx \Delta x^n \frac{\partial \rho(x, t)}{\partial t}
\] (15.19)

in the number of entities in time is given by the number of entities flowing into this volume via its surface, minus the number of entities flowing out. Starting with Eq. (2.31), this gives

\[
\frac{dN(x, t)}{dt} = -\oint_{\partial V} dA \cdot \left[ \rho(x, t) V(x, t) \right]
\]

\[
= -\int_{x_2 - \Delta x/2}^{x_2 + \Delta x/2} \cdots \int_{x_n - \Delta x/2}^{x_n + \Delta x/2} \rho(x) \left[ Q(x_1 + \Delta x/2, x_2, \ldots, x_n, t) \right.
\]

\[
- Q(x_1 - \Delta x/2, x_2, \ldots, x_n, t)] - \cdots
\]

\[
-\int_{x_1 - \Delta x/2}^{x_1 + \Delta x/2} \cdots \int_{x_{n-1} - \Delta x/2}^{x_{n-1} + \Delta x/2} \rho(x) \left[ Q(x_1, \ldots, x_{n-1}, x_n + \Delta x/2, t) \right.
\]

\[
- Q(x_1, \ldots, x_{n-1}, x_n - \Delta x/2, t)]
\]

\[
\approx -\Delta x^n \sum_{k=1}^{n} \frac{\partial Q_k(x, t)}{\partial x_k}.
\] (15.20)
where \( e_k \) are normalized unit vectors into the direction of coordinate axis \( k \). The approximate relationship becomes exact in the limit \( \Delta x \to 0 \). It directly implies the familiar continuity equation

\[
\frac{\partial \rho(x,t)}{\partial t} + \nabla_x \cdot [\rho(x,t)\mathbf{V}(x,t)] = 0,
\]

(15.21)

which can be rewritten as

\[
\frac{d\rho(x,t)}{dt} = \frac{\partial \rho(x,t)}{\partial t} + \mathbf{V}(x,t) \cdot \nabla_x \rho(x,t) = -\rho(x,t) \nabla_x \cdot \mathbf{V}(x,t)
\]

(15.22) \{totaln\}

by application of the product rule of Calculus [12].

**Exercise 2.8**

Let us derive the \( n \)-dimensional continuity equation directly from the equation of motion \( dx_i/dt = v_i \). We represent the location \( x_i(t) \) of an element \( i \) in space by a delta function \( \delta(x-x_i(t)) \) and introduce a normalized smoothing function \( s(||x'-x||) \) with

\[
\int_V d^n x' s(||x'-x||) = 1
\]

(15.23)

to define the local density

\[
\rho(x,t) = \int_V d^n x' s(||x'-x||) \sum_i \delta(x'-x_i(t)) = \sum_i s(||x_i(t)-x||).
\]

(15.24)

(15.25)

Herein, we sum up over all particles \( i \). We define the average velocity \( \mathbf{V}(x,t) \) as usual via a weighted average [with the weight function \( \delta(x-x_i(t))s(||x'-x||) \)]:

\[
\mathbf{V}(x,t) = \frac{\int_V d^n x' \sum_i v_i(t) \delta(x'-x_i(t)) s(||x'-x||)}{\int_V d^n x' \sum_i \delta(x-x_i(t)) s(||x'-x||)} = \frac{\int_V d^n x' \sum_i v_i(t) \delta(x'-x_i(t)) s(||x'-x||)}{\int_V d^n x' \sum_i \delta(x-x_i(t)) s(||x'-x||)}
\]

\[
= \frac{\sum_i v_i(t) s(||x_i(t)-x||)}{\rho(x,t)},
\]

(15.26)

which implies the flow formula

\[
\mathbf{Q}(x,t) = \rho(x,t)\mathbf{V}(x,t) = \sum_i v_i(t)s(||x_i(t)-x||).
\]

(15.27)

Differentiation of Eq. (15.24) with respect to time and application of the chain rule gives
\[
\frac{\partial \rho(x, t)}{\partial t} = \int_V d^n x' \sum_i \left( -\frac{dx_i}{dt} \right) \cdot \left[ \nabla_{x'} \delta(x' - x_i(t)) \right] s(||x' - x||) \\
= \int_V d^n x' \sum_i v_i(t) \delta(x' - x_i(t)) \left[ \nabla_{x'} s(||x' - x||) \right], \quad (15.28)
\]
where we have applied partial integration to obtain the last results (see Exercise ???), and considered the vanishing surface integral, as all entities are enclosed in the volume \( V \) (no entity shall be located on or passing its surface \( \partial V \)). Considering the symmetry of the smoothing function \( s(||x' - x||) \), we may replace \( \nabla_{x'} s(||x' - x||) \) by \(-\nabla_{x} s(||x' - x||)\), which finally gives
\[
\frac{\partial \rho(x, t)}{\partial t} = -\nabla_x \int_V d^n x' \sum_i v_i(t) \delta(x' - x_i(t)) s(||x' - x||) \\
= -\nabla_x \left[ \rho(x, t) V(x, t) \right]. \quad (15.29)
\]
To obtain this desired result, we have finally applied the definition (15.26) of the average velocity \( V(x, t) \). As a consequence of this, the validity of the continuity equation does not require an averaging over large numbers of entities, i.e. macroscopic volumes to average over. This makes the equation so powerful.

**Exercise 2.9**

Let us assume a closed system of volume \( V \). Integrating up the density \( \rho(x, t) \) over this volume gives the total number \( N(t) \) of elements:
\[
N(t) = \int_V d^n x \rho(x, t). \quad (15.30)
\]
We differentiate this formula with respect to time, insert the continuity equation (??), and evaluate the \( n \)-dimensional volume integral over \( V \), using the integral theorem of Gauß (see Exercise 6). In this way, we get
\[
\frac{dN(t)}{dt} = \int_V d^n x \frac{\partial \rho(x, t)}{\partial t} = -\int_V d^n x \nabla_x \cdot [\rho(x, t) V(x, t)] \quad (15.31)
\]
\[
= -\oint_{\partial V} dA \cdot [\rho(x, t) V(x, t)] = 0. \quad (15.32)
\]
The last step follows from the fact that, in a closed system, there can be no finite flow \( \mathbf{Q}(x, t) = \rho(x, t) \mathbf{V}(x, t) \) (15.33) through the surface \( \partial V \) of the system, i.e. at the boundary \( \partial V \) the flow \( \mathbf{Q}(x, t) \) is perpendicular to the surface element \( dA(x) \). In summary, we have \( dN(t)/dt = 0 \) in the volume \( V \) of a closed system, which implies a conservation of the number of elements.
Exercise 2.10

To construct a speed-density relationship \( V_\text{e}(\rho) \), for which the corresponding fundamental diagram \( Q_\text{e}(\rho) = \rho V_\text{e}(\rho) \) several maxima, let us start with a fundamental diagram given by a polynomial of fourth order:

\[
Q_\text{e}(\rho) = a_0 + a_1 \rho + a_2 \rho^2 + a_3 \rho^3 + a_4 \rho^4.
\]  

(15.34)

This function must fulfill several conditions so that we can interpret it as fundamental diagram. First, \( Q_\text{e}(0) = 0 \), i.e. \( a_0 = 0 \). Second, \( \frac{dQ_\text{e}(0)}{d\rho} > 0 \), i.e. \( a_1 > 0 \). Furthermore, we need to have three extrema for positive values \( \rho_k > 0 \) of \( \rho \): two maxima and one minimum. This implies that

\[
0 = \frac{dQ_\text{e}(\rho)}{d\rho} = a_1 + 2a_2 \rho + 3a_3 \rho^2 + 4a_4 \rho^3
\]

(15.35)

From this, we get the relationships

\[
\begin{align*}
a_1 & = -b \rho_1 \rho_2 \rho_3 > 0, \\
a_2 & = \frac{b}{2}(\rho_1 \rho_2 + \rho_2 \rho_3 + \rho_1 \rho_3), \\
a_3 & = -\frac{b}{3}(\rho_1 + \rho_2 + \rho_3), \\
a_4 & = \frac{b}{4}a_1.
\end{align*}
\]

(15.36)

In principle, we can choose any reasonable positive values for \( \rho_k \). For the minimum at \( \rho_2 \) with \( \rho_1 < \rho_2 < \rho_3 \), however, we must make sure that \( Q_\text{e}(\rho_2) > 0 \), i.e. the corresponding flow value is positive. Now, finally we should make sure that

\[
V_\text{e}(\rho) = \frac{Q_\text{e}(\rho)}{\rho} = a_1 + a_2 \rho + a_3 \rho^2 + a_4 \rho^3
\]

(15.37)

is monotonously decreasing, i.e.

\[
\frac{dV_\text{e}(\rho)}{d\rho} = a_2 + 2a_3 \rho + 3a_4 \rho^2 < 0,
\]

(15.38)

at least up to the density \( \rho = \rho_{\text{max}} > 0 \), for which \( Q_\text{e}(\rho_{\text{max}}) = 0 \). This implies \( a_2 < 0, a_3 < 0 \). Furthermore, \( a_3^2 < 3a_4a_2 \) and \( a_1 = V_\text{e}(0) > 0 \). An example is the speed-density relationship

\[
V_\text{e}(\rho) = \epsilon - \delta \rho + c(0.70 - \rho)^3
\]

(15.39)

with \( \delta = 0.1, \epsilon = 0.3, \) and \( c = 7.1 \).

Exercise 2.11

Let us start with the case of free traffic characterized by \( V_\text{e}(\rho) = Q_\text{e}(\rho)/\rho = V^0 \), and partially derive \( \rho(x, t) = \rho(x_0, t - (x - x_0)/V^0) \) with respect to \( x \), which gives

\[
\frac{\rho(x, t)}{\partial x} = \frac{\partial}{\partial x} \rho(x_0, t - (x - x_0)/V^0) = -\frac{1}{V^0} \frac{\partial}{\partial t} \rho(x_0, t - (x - x_0)/V^0) = -\frac{1}{V^0} \frac{\partial}{\partial t} \rho(x, t).
\]

(15.40)
Inserting this into the Lighthill-Whitham-Richards equation, we obtain
\[
\frac{\partial \rho(x, t)}{\partial t} + \frac{\partial}{\partial x} \left[ \rho V_e(\rho) \right] = \frac{\partial \rho(x, t)}{\partial t} + V_0 \frac{\partial \rho(x, t)}{\partial x} = 0, \tag{15.41}
\]
as demanded.

Let us now turn to the case of congested traffic characterized by
\[
\frac{d Q_e(\rho)}{d \rho} = c_0 = -|c_0|. 
\]
Partial derivation of
\[
\rho(x, t) = \rho(x_1, t - (x_1 - x)/|c_0|)
\]
with respect to \( x \) gives
\[
\rho(x, t) \frac{\partial}{\partial x} = \frac{1}{|c_0|} \frac{\partial}{\partial t} \rho(x_0, t - (x_1 - x)/|c_0|) = -\frac{1}{c_0} \frac{\partial \rho(x, t)}{\partial t}. \tag{15.42}
\]
Inserting this into the Lighthill-Whitham-Richards equation, we obtain
\[
\frac{\partial \rho(x, t)}{\partial t} + \frac{d Q_e(\rho)}{d \rho} \frac{\partial \rho(x, t)}{\partial x} = \frac{\partial \rho(x, t)}{\partial t} + c_0 \frac{\partial \rho(x, t)}{\partial x} = 0, \tag{15.43}
\]
as was to be shown.

Exercise 2.12

The model of Aw and Rascle can be rewritten in the form
\[
\frac{\partial V(x, t)}{\partial t} + V(x, t) \frac{\partial V(x, t)}{\partial x} = - \frac{d p(\rho)}{d \rho} \frac{\partial \rho(x, t)}{\partial t} - V(x, t) \frac{d p(\rho)}{d \rho} \frac{\partial \rho(x, t)}{\partial x}. \tag{15.44}
\]
Inserting the continuity equation for \( \partial \rho/\partial t \) on the right-hand side, we get
\[
\frac{\partial V(x, t)}{\partial t} + V(x, t) \frac{\partial V(x, t)}{\partial x} = \frac{d p(\rho)}{d \rho} \frac{\partial}{\partial x} \left[ \rho(x, t)V(x, t) \right] - V(x, t) \frac{d p(\rho)}{d \rho} \frac{\partial \rho(x, t)}{\partial x} \\
= \rho(x, t) \frac{d p(\rho)}{d \rho} \frac{\partial V(x, t)}{\partial x}. \tag{15.45}
\]
By comparison with the macroscopic velocity equation (2.59) we find
\[
- \frac{1}{\rho} \frac{\partial P}{\partial x} = \rho(x, t) \frac{d p(\rho)}{d \rho} \frac{\partial V(x, t)}{\partial x}. \tag{15.46}
\]
Assuming \( p(\rho) = \rho^\gamma \) implies \( d p(\rho)/d \rho = \gamma \rho^{\gamma - 1} \) and
\[
\frac{\partial P}{\partial x} = -\gamma \rho^\gamma \frac{\partial V}{\partial x} = \gamma \rho^{\gamma - 1} \frac{d p(x, t)}{d t}. \tag{15.47}
\]
Therefore, the traffic pressure according to the model of Aw and Rascle is a function of the velocity gradient rather than the density gradient. It is expected to grow, whenever the spatial gradient of the velocity along the road goes down. According to the continuity equation, this is directly related with an increase of the total time derivative \( d \rho/d t = (\partial \rho/\partial t + V \partial \rho/\partial x) \rho = -\rho \partial V/\partial x \) of the density, which reflects the density increase from the perspective of a car moving at velocity \( V \).
Exercise 2.13

The LWR model assumes by $V(x, t) = V_e(\rho(x, t))$ that the velocity $V$ is directly given by the density $\rho$. Let us define the distance $d_i(t)$ of a vehicle $i$ located at place $x_i(t)$ to the next vehicle ahead by $d_i(t) = 1/\rho(x_i(t), t)$ and $v_i(d_i) = V_e(1/d_i(t))$. Then,

$$v_i(t) = v_i(d_i(t)) = V_e(1/d_i(t))$$

(15.48)

constitutes the microscopic model which is equivalent to the LWR model.

Exercise 2.14

Determining the partial derivative of

$$G(x, t) = \frac{g_i(t) [x_{i-1}(t) - x] + g_{i-1}(t) [x - x_i(t)]}{x_{i-1}(t) - x_i(t)}$$

(15.49) \[\{\text{linint}\}\]

with respect to $x$ gives

$$\frac{\partial G(x, t)}{\partial x} = \frac{-g_i(t) + g_{i-1}(t)}{x_{i-1}(t) - x_i(t)}$$

(15.50)

for any specification of $g_i(t)$, e.g. $g_i(t) = v_i(t)$. The partial derivative with respect to time is

$$\frac{\partial G(x, t)}{\partial t} = \frac{dg_i(t)}{dt} \left[ x_{i-1}(t) - x \right] + g_i(t) \frac{dx_{i-1}(t)}{dt} + \frac{dg_{i-1}(t)}{dt} \left[ x - x_i(t) \right] - g_{i-1}(t) \frac{2x_i(t)}{dx_i(t)}$$

$$- \left( \frac{dx_{i-1}(t)}{dt} - \frac{dx_i(t)}{dt} \right) \left\{ \frac{g_i(t) [x_{i-1}(t) - x] + g_{i-1}(t) [x - x_i(t)]}{[x_{i-1}(t) - x_i(t)]^2} \right\}.$$  

(15.51)

For $g_i(t) = v_i(t) = dx_i(t)/dt$ and with $dv_i(t)/dt = f_i(t)$, this formula simplifies to the following expression:

$$\frac{\partial V(x, t)}{\partial t} = \frac{f_i(t) [x_{i-1}(t) - x] + v_i(t) v_{i-1}(t) + f_{i-1}(t) [x - x_i(t)] - v_{i-1}(t) v_i(t)}{x_{i-1}(t) - x_i(t)}$$

$$= \frac{\partial V(x, t)}{\partial x} \cdot V(x, t).$$

(15.52)

As a consequence, we find the exact relationship

$$\frac{\partial V(x, t)}{\partial t} + V(x, t) \frac{\partial V(x, t)}{\partial x} = A(x, t),$$

(15.53)

where one may specify the macroscopic acceleration $A(x, t)$ through the vehicle-based acceleration $a(v, \Delta v, s)$, but as a function of the macroscopic variables, i.e.

$$A(x, t) \approx a\left(V(x, t), \Delta V(x, t), S(x, t)\right)$$

(15.54)

with
\[ \Delta V(x, t) = \left[ V(x, t) - V(x + 1/\rho(x, t), t) \right] \approx -\frac{\partial V(x, t)}{\partial x} \frac{1}{\rho(x, t)} \] 

(15.55)

and

\[ S(x, t) = \frac{1}{2} \left( \frac{1}{\rho(x, t)} + \frac{1}{\rho(x + 1/\rho(x, t), t)} \right). \] 

(15.56)

Unfortunately, however, the expression for \( g_i(t) = 1/[x_{i-1}(t) - x_i(t)] \) does not simplify in a way that would finally lead to the continuity equation. Therefore, a micro-macro link based on the linear interpolation (15.49) of the microscopic variables \( g_i(t) \) does not imply the conservation of the number of vehicles, i.e. it is theoretically not consistent, even though it works well in practice [?].

**Exercise 2.15**

By differentiation of the velocity formula and consideration of the symmetry of the smoothing function we get

\[ \frac{\partial}{\partial t} [\rho(x, t)V(x, t)] = \sum_i \frac{dv_i(t)}{dt} s(x_i(t) - x) + \sum_i v_i(t) \frac{\partial}{\partial x_i} \left[ s(x_i(t) - x) \right] \frac{dx_i(t)}{dt} \]

\[ = \sum_i f_i(t) s(x_i(t) - x) - \sum_i v_i(t)^2 \frac{\partial}{\partial x} \left[ s^2(x_i(t) - x) \right]. \] 

(15.57)

Taking into account

\[ \sum_i [v_i(t)]^2 \frac{\partial}{\partial x} \left[ s(x_i(t) - x) \right] \]

\[ = \sum_i \left\{ [V(x, t)]^2 + 2V(x, t)\delta s_i(t) + [\delta s_i(t)]^2 \right\} \frac{\partial}{\partial x} \left[ s(x_i(t) - x) \right] \]

\[ = \rho(x, t)[V(x, t)]^2 + 0 + \rho(x, t)\theta(x, t). \] 

(15.58)

Altogether, we have

\[ \frac{\partial}{\partial t} [\rho(x, t)V(x, t)] = -\frac{\partial}{\partial x} \left[ \rho(x, t) [V(x, t)^2 + \theta(x, t)] \right] + \sum_i f_i(t) s(x_i(t) - x). \] 

(15.59)

Now, we carry out the partial differentiation applying the product rule of Calculus to obtain

\[ \rho(x, t) \frac{\partial V(x, t)}{\partial t} = -V(x, t) \frac{\partial \rho(x, t)}{\partial t} - V(x, t) \frac{\partial}{\partial x} [\rho(x, t)V(x, t)] \]

\[ = -\rho(x, t)V(x, t) \frac{\partial V(x, t)}{\partial x} - \frac{\partial}{\partial x} [\rho(x, t)\theta(x, t)] + \sum_i f_i(t) s(x_i(t) - x)]. \] 

(15.60)

Inserting the continuity equation \( \partial \rho/\partial t = -\partial /\partial x (\rho V) \) and dividing the above equation by \( \rho(x, t) \) finally gives the velocity equation

\[ \frac{\partial V(x, t)}{\partial t} + V(x, t) \frac{\partial V(x, t)}{\partial x} = - \frac{1}{\rho(x, t)} \frac{\partial}{\partial x} [\rho(x, t)\theta(x, t)] + \frac{1}{\rho(x, t)} \sum_i f_i(t) s(x_i(t) - x)]. \] 

(15.61)
which was to be shown. If the smoothing function is specified by
\[ s(x_i - x) = \frac{\varrho}{2} \begin{cases} 1 & \text{if } |x_i - x| \leq 1/\varrho \\ 0 & \text{otherwise}, \end{cases} \tag{15.62} \]
with a large smoothing window \( \Delta x = 1/\varrho \), the number of vehicles \( i \) within the smoothing interval \([x - 1/\varrho, x + 1/\varrho]\) is expected to be \( 2\rho/\varrho \), where \( \rho \) represents the average vehicle density in this interval. Therefore,
\[ \rho(x, t) = \sum_i s(x_i(t) - x) = \frac{2\rho}{\varrho} \frac{\varrho}{2} = \rho. \tag{15.63} \]
Therefore, smoothing over larger spatial intervals still gives theoretically consistent results. In the following, we will use the specification
\[ \varrho = \frac{1}{x_k - x_{k+1}} \text{ for } x_{k+1} < x \leq x_k, \tag{15.64} \]
which implies
\[ \sum_i s(|x_i(t) - x|) f_{i,i-1}(t) = \frac{\rho(x,t)}{2} f \left( \frac{1}{\rho(x,t)} \right) + \frac{\rho(x,t)}{2} f \left( \frac{1}{\rho(x+1/\rho,t)} \right) \tag{15.65} \] {using1}
according to Eqs. (??) through (??).

Exercise 2.16

A Taylor approximation of first order gives
\[ f \left( \frac{1}{\rho(x+1/\rho,t)} \right) \approx f \left( \rho(x,t) + \frac{1}{\rho(x,t)} \frac{\partial \rho(x,t)}{\partial x} \right) \]
\[ \approx f \left( \frac{1}{\rho(x,t)} \left( 1 - \frac{\partial \rho(x,t)}{\partial x} \frac{1}{\rho(x,t)^2} \right) \right) \]
\[ \approx f \left( \frac{1}{\rho(x,t)} + \frac{df(d)}{dd} \cdot \left( -\frac{\partial \rho(x,t)}{\partial x} \frac{1}{\rho(x,t)^3} \right) \right), \tag{15.66} \]
where we have applied the geometric series expansion \( 1/(1-z) \approx 1 + z + \ldots \). (For a higher order derivation see Ref. [??].) Note that the definition \( V_\epsilon(\rho) = V^0 + \tau f(1/\rho) = V^0 + \tau f(\rho) \) and \( \rho = 1/d \) imply
\[ \frac{df(d)}{dd} = \left( \frac{d}{d\rho} \frac{V_\epsilon(\rho) - V^0}{\tau} \right) \frac{d\rho}{dd} = \frac{1}{\tau} \frac{dV_\epsilon(\rho)}{d\rho} \cdot \left( -\frac{1}{d^2} \right) = -\frac{\rho^2}{d^2} \frac{dV_\epsilon(\rho)}{d\rho}. \tag{15.67} \]
Therefore, using Eq. (15.65), we finally obtain:
\[ \sum_i s(|x_i(t) - x|) f_{i,i-1}(t) \approx \rho(x,t) f \left( \frac{1}{\rho(x,t)} \right) + \frac{1}{2\tau} \frac{dV_\epsilon(\rho)}{d\rho} \frac{\partial \rho(x,t)}{\partial x}. \tag{15.68} \]
In comparison with the zeroth order approximation result (2.82) derived in Sec. 2.7, in first order Taylor approximation we find the following additional contribution on the right-hand side of the macroscopic velocity equation:
15 Problem Solutions

\[ \frac{1}{2\tau(x,t)} \frac{dV_c(\rho)}{d\rho} \frac{\partial p(x,t)}{\partial x} \approx \frac{1}{2\tau(x,t)} \frac{\partial V(x,t)}{\partial x}. \]  

(15.69) \{zuspress\}

Notably enough, if the velocity variance \( \theta \) is zero, this contribution corresponds exactly to the pressure term \( P = [V^0 - V_c(\rho)]/(2\tau) \) in Payne’s macroscopic traffic model, see Eq. (2.62). It also reminds of the pressure term in the traffic model of Aw and Rascle (see Exercise 2.12). In the general case, i.e. for a non-vanishing velocity variance \( \theta \neq 0 \), we expect a pressure term

\[ P(x,t) = \rho \theta(x,t) + \frac{V^0(x,t) - V_c(\rho(x,t))}{2\tau}, \]  

(15.70)

see Eqs. (??) and (??).

Exercises of Chapter 3

NOTE: EXERCISES IN THIS SECTION NOT YET FULLY CHECKED AND COMPLETED

Exercise 3.1

First, let us consider the continuity equation, where we have \( \rho(x,t) = \delta \rho(x,t) + \rho_c, V(x,t) = \delta V(x,t) + V_c \). Insert this into the continuity equation, we get

\[ \frac{\partial \delta \rho(x,t)}{\partial t} + (V_c + \delta V(x,t)) \frac{\partial \delta \rho(x,t)}{\partial x} + \rho_c \frac{\partial \delta V(x,t)}{\partial x} = -\rho \frac{\partial p(x,t)}{\partial x}. \]  

(15.71)

Now, we can separate the second order terms from the Eq. (1.1), and move them to the right side of the equation.

\[ \frac{\partial \delta \rho(x,t)}{\partial t} + V_c \frac{\partial \delta \rho(x,t)}{\partial x} + \rho_c \frac{\partial \delta V(x,t)}{\partial x} = -\delta V(x,t) \frac{\partial \delta \rho(x,t)}{\partial x} - \delta \rho(x,t) \frac{\partial V(x,t)}{\partial x}. \]  

(15.72)

Neglect the second order terms, we obtain the linearized continuity equation

\[ \frac{\partial \delta \rho(x,t)}{\partial t} + V_c \frac{\partial \delta \rho(x,t)}{\partial x} + \rho_c \frac{\partial \delta V(x,t)}{\partial x} = 0. \]  

(15.73)

Similarly, we insert \( \rho(x,t) = \delta \rho(x,t) + \rho_c, V(x,t) = \delta V(x,t) + V_c \) into the velocity equation, and have

\[ \frac{\partial \delta V(x,t)}{\partial t} + (V_c + \delta V(x,t)) \frac{\partial \delta V(x,t)}{\partial x} = -\frac{1}{\rho_c} \frac{1}{1 + \delta \rho/\rho_c} \frac{\partial P(\delta \rho + \rho_c)}{\partial x}. \]  

(15.74)

Applying Taylor expansion to \( P(\delta \rho + \rho_c) \) and \( 1/(1 + \delta \rho/\rho_c) \), we have

\[ P(\delta \rho + \rho_c) = P(\rho_c) + \frac{dP(\rho_c)}{d\rho} \delta \rho + o(\delta \rho)^2 + \ldots \]  

(15.75)
Similarly, we get

\begin{align}
\frac{1}{1 + \delta \rho / \rho_e} = 1 - \delta \rho / \rho_e + o(\delta \rho)^2 + \ldots \quad (15.76)
\end{align}

Insert Eq. (1.5???) and (1.6???) into (1.4???), and only keep the first order term, gives

\begin{align}
\frac{\partial \delta V(x,t)}{\partial t} + V_e \frac{\partial \delta V(x,t)}{\partial x} = -\frac{1}{\rho_e} (1 - \frac{\delta \rho}{\rho_e}) \frac{dP(\rho_e)}{d\rho} \frac{\partial \delta \rho(x,t)}{\partial x}.
\end{align}

Again, we remove the second order terms, thus the linearized velocity equation is as follow

\begin{align}
\frac{\partial \delta V(x,t)}{\partial t} + V_e \frac{\partial \delta V(x,t)}{\partial x} = -\frac{1}{\rho_e} \frac{dP(\rho_e)}{d\rho} \frac{\partial \delta \rho(x,t)}{\partial x}
\end{align}

**Exercise 3.2**

In order to check the general solution of waves equation, we assume that

\begin{align}
x' = x - c \partial t,
\end{align}

\begin{align}
x'' = x + c \partial t.
\end{align}

Then, we have \( \delta \rho(x,t) = F(x') + G(x'') \). Insert this into the waves equation, we have

\begin{align}
\frac{\partial^2 \delta \rho(x,t)}{\partial t^2} - c^2 \frac{\partial^2 \delta \rho(x,t)}{\partial x'^2} - c^2 \frac{\partial^2 \delta \rho(x,t)}{\partial x''^2} = (15.81)
\end{align}

Now let us consider the special solution with the initial condition \( \delta \rho(x,0) = \exp(-x^2), \partial \delta \rho(x,0)/\partial t = 0 \).

\begin{align}
F(x) + G(x) = \exp(-x^2),
\end{align}

\begin{align}
-cF'(x) + cG'(x) = 0.
\end{align}

Inserting Eq. (1.15???) into (1.16???), we get

\begin{align}
F'(x) = -x \exp(-x^2),
\end{align}

\begin{align}
F(x) = \int -x \exp(-x^2) dx,
\end{align}

\begin{align}
= \frac{1}{2} \exp(-x^2) + C_0.
\end{align}

Similarly, we get \( G(x) = \exp(-x^2)/2 + C_1 \). Therefore, the special solution is
\[ \delta \rho(x, t) = F(x - \dot{c}t) + G(x + \dot{c}t) \]  
\[ = \frac{1}{2} \exp(-(x - \dot{c}t)^2) + \frac{1}{2} \exp(-(x + \dot{c}t)^2) + C'. \]  
(15.90)

Apply the initial condition again gives \( C' = 0 \).

With the initial condition \( \delta \rho(x, t) = 0, \partial \delta \rho(x, 0)/\partial t = \cos(x) \), we obtain

\[ F(x) + G(x) = 0, \]  
(15.92)

\[ -\dot{c}F'(x) + \dot{c}G'(x) = \cos(x). \]  
(15.93)

Solving the equations, we get

\[ F(x) = -\frac{1}{2\dot{c}} \sin(x) - C_0, \]  
(15.94)

\[ G(x) = \frac{1}{2\dot{c}} \sin(x) + C_0. \]  
(15.95)

Finally,

\[ F(x - \dot{c}t) = -\frac{1}{2\dot{c}} \sin(x - \dot{c}t) - C_0, \]  
(15.96)

\[ G(x + \dot{c}t) = \frac{1}{2\dot{c}} \sin(x + \dot{c}t) + C_0, \]  
(15.97)

\[ \delta \rho(x, t) = -\frac{1}{2\dot{c}} \sin(x - \dot{c}t) + \frac{1}{2\dot{c}} \sin(x + \dot{c}t). \]  
(15.98)

**Exercise 3.3**

Let us start with the change of variables

\[ x'(t) = x - \int_0^t dt' c(y_x(t'), t'), \]  
(15.99)

we have

\[ \rho(x, t) = \rho(x - \int_0^t dt' c(y_x(t'), t')), 0) = \rho(x', 0). \]  
(15.100)

Differentiate \( \rho(x', 0) \) partially with respect to \( t \) gives

\[ \frac{\partial \rho(x', 0)}{\partial t} = \frac{\partial x'}{\partial t} \frac{\partial \rho(x', 0)}{\partial x'} = -c(\rho(y_x(t), t)) \frac{\partial \rho(x', 0)}{\partial x'}. \]  
(15.101)

Here,

\[ y_x(t) = x - \int_0^t dtv(t) = x, \]  
(15.102)

so \( c(\rho(y_x(t), t)) = c(\rho(x, t)) \). Therefore, Eq. (1.31???) could be rewritten as

\[ \frac{\partial \rho(x', 0)}{\partial t} = -c(\rho(x, t)) \frac{\partial \rho(x', 0)}{\partial x'}. \]  
(15.103)

We now differentiate \( \rho(x', 0) \) partially with respect to \( x \) gives
\[ \frac{\partial \rho(x', 0)}{\partial x} = \frac{\partial \rho(x', 0)}{\partial x'} \frac{\partial x'}{\partial x} = \frac{\partial \rho(x', 0)}{\partial x'}. \] (15.104)

Insert Eq. (1.34?) into (1.33?), we obtain
\[ \frac{\partial \rho(x', 0)}{\partial t} + c(\rho(x, t)) \frac{\partial \rho(x', 0)}{\partial x} = 0. \] (15.105)

Considering the case, when \( c(\rho) = c \equiv \text{const} \), we have a simple analytical solution
\[ \rho(x, t) = \rho(x_0 - V_0^0 t, 0), \] (15.106)
for \( x > x_0 \) in the free flow regime with \( \rho(x, t) < \rho_c \), and
\[ \rho(x, t) = \rho(x_1 - c_1 t, 0), \] (15.107)
for \( x < x_1 \) in the congested flow regime with \( \rho(x, t) > \rho_c \).

For the free flow regime, the characteristic speed is determined by
\[ c(\rho) = V_0^0 = \frac{dQ(\rho)}{d\rho}, \] (15.108)
\[ Q(\rho) = V_0^0 \rho + C_0. \] (15.109)

Since it is obvious that the flow is always zero, when the density is zero, i.e. \( Q(0) = 0 \), therefore we have
\[ Q(\rho) = V_0^0 \rho. \] (15.110)

For the congested flow regime, we use the similar process
\[ c(\rho) = c_0 = \frac{dQ(\rho)}{d\rho}, \] (15.111)
\[ Q(\rho) = c_0 \rho + C'_0, \] (15.112)
\[ V_c(\rho) = c_0 + \frac{C'_0}{\rho}. \] (15.113)

When the maximum density is reached in the road, the equilibrium velocity is reduced to zero, i.e. \( V_c(\rho_{\text{max}}) = 0 \). With this condition, the coefficient is obtained as
\[ C'_0 = -c_0 \rho_{\text{max}}. \] (15.114)

Insert Eq. (1.44???) into (1.42???), we have
\[ Q(\rho) = -c_0 \rho_{\text{max}} \left(1 - \frac{\rho}{\rho_{\text{max}}}\right). \] (15.115)

**Exercise 3.4**

Assuming that \( s \) is the velocity of shock wave, which starts at the origin of the axis. The number of vehicles could be given by
\[ \int_{-M}^{M} \rho dx = (M + st)\rho_- + (M - st)\rho_+. \] (15.116)
here $M$ is large compared to $s$. Therefore, we have

$$
\frac{d}{dt} \int_{-M}^{M} \rho dx = s(\rho_- - \rho_+).
$$

(15.117)

According to the conservation law of the number of vehicles, we get

$$
\frac{d}{dt} \int_{-M}^{M} \rho dx = f(\rho_-) - f(\rho_+).
$$

(15.118)

Compare (1.47) and (1.48) gives

$$
f(\rho_-) - f(\rho_+) = s(\rho_- - \rho_+).
$$

(15.119)

This gives simply

$$
s = \frac{f(\rho_-) - f(\rho_+)}{(\rho_- - \rho_+)}. 
$$

(15.120)

---

**DERIVATION OF THE SPEED OF SHOCK WAVES FROM THE CONSERVATION OF THE NUMBER OF VEHICLES**

The formula for the shock wave velocity follows from the conservation of the number of vehicles: Let us assume the shock front is located at position $x$. The high-density area downstream of $x$ moves upstream by a distance $\Delta x$ during a time interval $\Delta t$ that is needed for $(\rho_+ - \rho_-) \Delta x$ vehicles to join the queue downstream of $x$. The number of vehicles accumulating between locations $x - \Delta x$ and $x$, however, is given by the difference $[Q_e(\rho_+) - Q_e(\rho_-)]$ of the inflow and the outflow, multiplied by the time interval $\Delta t$. As the propagation speed $C(\rho_-, \rho_+)$ is given by $\Delta x/\Delta t$, we end up with Eq. (3.20).

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**Exercises of Chapter 4**

NOTE: EXERCISES IN THIS SECTION NOT YET FULLY CHECKED AND COMPLETED

**Exercise 4.1**

Let us start from the partial differentiation of the form of density with respect to $t$.

$$
\frac{\partial \rho(x,t)}{\partial t} = \frac{\partial}{\partial t} \left( \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} \rho(x',0) \exp\left(-\frac{(x - x')^2}{4Dt}\right) dx' \right)
$$

(15.121)

$$
= \left(-\frac{1}{2t}\right) \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} \rho(x',0) \exp\left(-\frac{(x - x')^2}{4Dt}\right) dx'
$$

(15.122)

$$
+ \frac{(x - x')^2}{4Dt} \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} \rho(x',0) \exp\left(-\frac{(x - x')^2}{4Dt}\right) dx'
$$

(15.123)

Now differentiate partially with respect to $x$, we have
\[
D \frac{\partial^2 \rho(x,t)}{\partial x^2} = D \frac{\partial}{\partial x} \left( \frac{1}{\sqrt{4\pi D t}} \int_{-\infty}^{\infty} \rho(x',0) \exp\left(-\frac{(x-x')^2}{4Dt}\right) dx' \right) 
\]

(15.124)

\[
= -D \frac{\partial}{\partial x} \left( \frac{1}{2Dt} \frac{1}{\sqrt{4\pi D t}} \int_{-\infty}^{\infty} \rho(x',0) \exp\left(-\frac{(x-x')^2}{4Dt}\right) dx' \right) 
\]

(15.125)

\[
= -\frac{1}{2t} \frac{1}{\sqrt{4\pi D t}} \int_{-\infty}^{\infty} \rho(x',0) \exp\left(-\frac{(x-x')^2}{4Dt}\right) dx' 
\]

(15.126)

\[
+ \frac{(x-x')^2}{4Dt^2} \frac{1}{\sqrt{4\pi D t}} \int_{-\infty}^{\infty} \rho(x',0) \exp\left(-\frac{(x-x')^2}{4Dt}\right) dx'. 
\]

(15.127)

Compare Eq.(1.7???) with (1.3???), obviously the form of density is a solution to the diffusion equation.

Assuming that all elements are initially concentrated in one location \(x = x_0\), we have

\[
\rho(x,t) = \frac{1}{\sqrt{4\pi D t}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-x_0)^2}{4Dt}\right) dx
\]

(15.128)

\[
\int_{-\infty}^{\infty} dx (x-x_0)^2 \rho(x,t)
\]

(15.129)

\[
= \frac{1}{\sqrt{4\pi D t}} \int_{-\infty}^{\infty} dx (x-x_0)^2 \exp\left(-\frac{(x-x_0)^2}{4Dt}\right).
\]

(15.130)

We may change the variable by \(y = x - x_0\), and have

\[
\frac{1}{\sqrt{4\pi D t}} \int_{-\infty}^{\infty} y^2 e^{-y^2 / 4Dt} dy.
\]

(15.131)

Let us consider the following integral of exponential equation

\[
I(\alpha) = \int_{-\infty}^{\infty} e^{-\alpha x^2} dx,
\]

(15.132)

(15.133)

where \(\alpha > 0\).

Taking the square of \(I(\alpha)\),

\[
I^2(\alpha) = \int_{-\infty}^{\infty} e^{-\alpha x^2} dx \cdot \int_{-\infty}^{\infty} e^{-\alpha y^2} dy 
\]

(15.134)

\[
= \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} e^{-\alpha y^2} dy \right) e^{-\alpha x^2} dx 
\]

(15.135)

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\alpha(x^2+y^2)} dxdy 
\]

(15.136)

Switching from cartesian coordinates to polar coordinates gives

\[
x = r \cos \theta, 
\]

(15.137)

\[
y = r \sin \theta, 
\]

(15.138)

\[
dxdy = rd\theta dr
\]

(15.139)

Therefore, we get
\[ I^2(\alpha) = \int_0^{2\pi} \int_0^\infty re^{-\alpha r^2} drd\theta \] (15.140)
\[ = 2\pi \int_0^\infty \frac{1}{2} e^{-\alpha r^2} dr^2 \] (15.141)
\[ = \frac{\pi}{-\alpha} e^{-\alpha r^2} \bigg|_0^\infty \] (15.142)
\[ = \frac{\pi}{\alpha} \] (15.143)

We obtain
\[ I(\alpha) = \sqrt{\frac{\pi}{\alpha}}. \] (15.144)

Differentiate \( I(\alpha) \) with respective to \( \alpha \) gives
\[ \frac{dI(\alpha)}{d\alpha} = \int_{-\infty}^{\infty} -x^2 e^{-\alpha x^2} dx \] (15.145)
\[ = -\frac{1}{2\alpha} \sqrt{\frac{\pi}{\alpha}}. \] (15.146)

We now have
\[ \int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} dx \] (15.147)
\[ = \frac{1}{2\alpha} \sqrt{\frac{\pi}{\alpha}}. \] (15.148)

Applying the above formula to Eq.(1.4???) gives us
\[ \frac{1}{\sqrt{4\pi Dt}} \int_{-\infty}^{\infty} y^2 e^{-y^2/4Dt} dy \] (15.149)
\[ = \frac{1}{\sqrt{4\pi Dt}} \frac{4Dt}{2} \sqrt{4Dt\pi} \] (15.150)
\[ = 2Dt. \] (15.151)

Exercise 4.2

By the differentiation the \( \rho(x, t) \) with respect to \( t \), gives
\[ \frac{\partial \rho(x, t)}{\partial t} = \frac{g'(t)}{2g(t)} \rho(x, t) + \frac{1}{g(t)} \left[ \frac{(x - f(t))f'(t)}{g(t)} + \frac{|x - f(t)|^2}{2g'(t)} \right] \rho(x, t) \] (15.152)

Applying the product, quotient, and chain rules of differential calculus, we find
\[ \frac{\partial^2 \rho(x, t)}{\partial^2 x} = \frac{\partial}{\partial x} \left[ \frac{f(t) - x}{g(t)} \right] \rho(x, t) \] (15.153)
\[ = \frac{f(t)(f(t) - x) - g(t) - x(f(t) - x)}{g^2(t)} \rho(x, t) \] (15.154)
\[ = -\frac{1}{g(t)} \rho(x, t) + \frac{|x - f(t)|^2}{g(t)^2} \rho(x, t). \] (15.155)
Comparing the expression for $\partial P(x, t)/\partial t$ with the one for $\partial^2 P(x, t)/\partial x^2$ implies that
\[
\frac{\partial \rho(x, t)}{\partial t} = \frac{g'(t)}{2} \frac{\partial^2 \rho(x, t)}{\partial x^2} + \frac{|x - f(t)| f'(t)}{g(t)} \rho(x, t). \tag{15.156}
\]

Assuming that $D$ is a constant, we obtain
\[
g(t) = g(0) + 2Dt, \tag{15.157}
\]
\[
f(t) = f(0) + \lambda g(0)t + \lambda Dt^2. \tag{15.158}
\]

However, for $D(t) = C_0g(t)$, we would find an exponential growth
\[
g(t) = g(0)e^{2C_0t}, \tag{15.159}
\]
\[
f(t) = (f(0) - \frac{\lambda g(0)}{2C_0}) + \frac{\lambda g(0)}{2C_0} e^{2C_0t}. \tag{15.160}
\]

Finally, for $D(t) = C_1f(t)$, we have
\[
f(t) = f(0) \cosh(\sqrt{2\lambda C_1}t) + \frac{\sqrt{\lambda} g(0)}{\sqrt{2C_1}} \sinh(\sqrt{2\lambda C_1}t). \tag{15.161}
\]

**Exercise 4.3**

Let us start from Eq.(4.9)
\[
\frac{\partial c(x, t)}{\partial t} + c(x, t) \frac{\partial c(x, t)}{\partial x} = D \frac{\partial^2 c(x, t)}{\partial x^2}, \tag{15.162}
\]
into which we insert the following equation
\[
c(x, t) = \frac{\partial \psi'(x, t)}{\partial x}. \tag{15.163}
\]

We then have
\[
\frac{\partial^2 \psi'}{\partial x^2} + \frac{\partial \psi'}{\partial x} \frac{\partial^2 \psi'}{\partial x^2} = D \frac{\partial^3 \psi'}{\partial x^3} \tag{15.164}
\]
\[
= \frac{\partial^2 \psi'}{\partial x dt} + \frac{1}{2} \frac{\partial}{\partial x} (\frac{\partial \psi'}{\partial x})^2 = D \frac{\partial^3 \psi'}{\partial x^3} \tag{15.165}
\]

Integrate over $x$, we obtain
\[
\frac{\partial \psi'}{\partial t} + \frac{1}{2} \left( \frac{\partial \psi'}{\partial x} \right)^2 = D \frac{\partial^2 \psi'}{\partial x^2}. \tag{15.166}
\]

Let us apply the follow transformation
\[
\psi'(x, t) = -2D \ln \psi(x, t), \tag{15.167}
\]
and have
\[
\frac{\partial}{\partial t} \left[ -2D \ln \psi(x,t) \right] + \frac{1}{2} \left( \frac{\partial}{\partial x} \left[ -2D \ln \psi(x,t) \right] \right)^2 \quad (15.168)
\]

\[
= D \frac{\partial}{\partial x} \left\{ \frac{\partial}{\partial x} \left[ -2D \ln \psi(x,t) \right] \right\} \quad (15.169)
\]

\[
= -2D \frac{1}{\psi(x,t)} \frac{\partial \psi(x,t)}{\partial t} + \frac{1}{2} \left[ -2D \frac{1}{\psi(x,t)} \frac{\partial \psi(x,t)}{\partial x} \right]^2 \quad (15.170)
\]

\[
= D \frac{\partial}{\partial x} \left\{ -2D \frac{1}{\psi(x,t)} \frac{\partial \psi(x,t)}{\partial x} \right\} \quad (15.171)
\]

\[
= -2D \frac{1}{\psi(x,t)} \frac{\partial \psi(x,t)}{\partial x} + 2D^2 \frac{1}{\psi(x,t)^2} \left( \frac{\partial \psi(x,t)}{\partial x} \right)^2 \quad (15.172)
\]

\[
= -2D^2 \frac{1}{\psi(x,t)} \frac{\partial^2 \psi(x,t)}{\partial x^2} + 2D^2 \frac{1}{\psi(x,t)^2} \left( \frac{\partial \psi(x,t)}{\partial x} \right)^2 \quad (15.173)
\]

Finally, we have

\[
\frac{\partial \psi(x,t)}{\partial t} = D \frac{\partial^2 \psi(x,t)}{\partial x^2} . \quad (15.174)
\]

**Exercise 4.4**

The Riemann-Liouville approach to definition of fractional integral is given by

\[
J^\alpha f(t) = f_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau , \quad (15.175)
\]

where \( t > 0, \alpha \in \mathbb{R}^+ \).

Introducing the positive integer \( m \) such that \( m - 1 < \alpha \leq m \), one defines the Fractional Derivative of order \( \alpha > 0 \) as \( D^\alpha f(t) = D^m J^{m-\alpha} f(t) \), namely

\[
D^\alpha f(t) = \frac{d^m}{dt^m} \left[ \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau \right] \quad (15.176)
\]

Defining for complementation \( D^0 = J^0 = I \), then we have \( D^\alpha J^\alpha = I \). Consider the equation of heat flow

\[
u_t - u_{xx} = 0, u = u(x,t), \quad (15.177)
\]

in the semi-infinite intervals \( 0 < x < \infty \) and \( 0 < t < \infty \) of space and time, respectively. Here the dimensionless equation \( u = u(x,t) \) means temperature. Assume vanishing initial temperature, i.e. \( u(x,0) = 0 \) for \( 0 < x < \infty \) and given influx across the boundary \( x = 0 \) from \( x < 0 \) to \( x > 0 \),

\[-u_x(0,t) = p(t) . \quad (15.178)\]

Using the approach of fractional calculus presented above to solve the inverse boundary value problem.
We now use the special solution of diffusion equation, and have

\[
u(x, t) = \frac{1}{\sqrt{\pi}} \int_0^t \frac{p(\tau)}{\sqrt{t - \tau}} e^{-x^2/(4(t - \tau))} d\tau, \quad (15.179)
\]

Considering the interior boundary temperature \(\phi(t) = u(0^+, t), t > 0\), which could be represented as

\[
\frac{1}{\sqrt{\pi}} \int_0^t \frac{p(\tau)}{\sqrt{t - \tau}} d\tau = J_{1/2} \tau p(t) = \phi(t).
\]

The above equation is an Abel integral equation of first kind for determination of an unknown influx \(p(t)\) if the interior boundary temperature \(\phi(t)\) is given by measurements or intended to be achieved by controlling the influx. With \(m = 1, \alpha = 1/2\), as

\[
p(t) = D^{1/2} \phi(t) = \frac{1}{\sqrt{\pi}} \frac{d}{dt} \int_0^t \frac{\phi(\tau)}{\sqrt{t - \tau}} d\tau.
\]

Considering the following special cases,

\[
(i) \phi(t) = t, p(t) = \frac{1}{2} \sqrt{\pi} t, \quad (15.182)
\]

\[
(ii) \phi(t) = 1, p(t) = \frac{1}{\sqrt{\pi} t}
\]

Therefore, for linear increase of interior boundary temperature the required influx is continuous and increasing from 0 towards \(\infty\) (with unbounded derivative at \(t = 0^+\)), whereas for instantaneous jump-like increase from 0 to 1 the required influx decreases from \(\infty\) at \(t = 0^+\) to 0 as \(t \to \infty\).

Exercises of Chapter 5

Exercise 5.1

Let us start with the common representation of a complex numbers

\[
z = |z|e^{\pm i\varphi} = |z| \left[ \cos(\varphi) \pm i \sin(\varphi) \right], \quad (15.184)
\]

which implies

\[
z^2 = |z|^2 \left[ \cos^2(\varphi) \pm 2i \cos(\varphi) \sin(\varphi) - \sin^2(\varphi) \right]. \quad (15.185)
\]

On the other hand, using the well-known law \(e^{x_1} \cdot e^{x_2} = e^{x_1+x_2}\) for the exponential function, we find the alternative formula

\[
z^2 = |z|^2 \left[ e^{\pm i\varphi} \right]^2 = |z|^2 e^{\mp 2i\varphi} = |z|^2 \left[ \cos(2\varphi) \pm i \sin(2\varphi) \right]. \quad (15.186)
\]

Comparing the real parts and using the trigonometric relationship \(\sin^2(x) + \cos^2(x) = 1\), we find
\[ \cos(2\varphi) = 1 - 2 \sin^2(\varphi) = 1 - 2[1 - \cos^2(\varphi)] = 2 \cos^2(\varphi) - 1, \]  
(15.187)

from which we can derive the trigonometric formulas

\[ \sin^2(\varphi/2) = \frac{1}{2} [1 - \cos(\varphi)] \quad \text{and} \quad \cos^2(\varphi/2) = \frac{1}{2} [1 + \cos(\varphi)]. \]  
(15.188)

Therefore, the square root of a complex number is given by

\[ \sqrt{z} = \sqrt{|z| e^{i \varphi}} = \sqrt{|z|} e^{i \varphi/2} = \sqrt{|z|} \cos(\varphi/2) \pm i \sin(\varphi/2). \]  
(15.189)

Exercise 5.2

Let us start with the representation (2.52) of the Lighthill-Whitham-Richards model. The linearized model equation can then simply be written as

\[ \frac{\partial \delta \rho}{\partial t} + c(\rho_e) \frac{\partial \delta \rho}{\partial x} = 0. \]  
(15.191)

Inserting the ansatz (5.11), we obtain

\[ (\lambda - i\omega) \delta \rho(x, t) + i\kappa c(\rho_e) \delta \rho(x, t) = 0. \]  
(15.192)

This implies

\[ \lambda = 0 \quad \text{and} \quad \omega = \kappa c(\rho). \]  
(15.193)

As a consequence of \( \lambda = \text{Re} \lambda = 0 \), the linearized continuity equation can never behave unstable. In fact, it is marginally stable, i.e. the amplitude neither grows nor decays. This independently confirms our conclusions drawn in Sec. 3.2. Adding the linearized diffusion term \( D\frac{\partial^2 \delta \rho(x, t)}{\partial x^2} \) on the right-hand side of Eq. (15.191) results in \( \lambda(\kappa) = -\kappa^2 D \), i.e. the Burgers equation will always behave stable with respect to small perturbations.

Exercise 5.3

The linearized continuity equation remains as in Eq. (15.191), while the linearized velocity equation becomes

\[ \frac{\partial \delta V}{\partial t} + V \frac{\partial \delta V}{\partial x} = -\frac{1}{\rho_e} \left[ \frac{\partial P_e(\rho_e, V_e)}{\partial \rho} \frac{\partial \delta \rho}{\partial x} + \frac{\partial P_e(\rho_e, V_e)}{\partial V} \frac{\partial \delta V}{\partial x} \right] + \frac{1}{7} \left[ \frac{\partial V_e(\rho_e, V_e)}{\partial \rho} \delta \rho(x, t) + \frac{\partial V_e(\rho_e, V_e)}{\partial V} \delta V(x, t) - \delta V(x, t) \right]. \]  
(15.194)

With the ansatz (5.11), we obtain the eigenvalue problem:
The corresponding characteristic polynomial reads
\[
(\tilde{\lambda})^2 + \tilde{\lambda} \left[ \frac{i\kappa}{\rho_e} \frac{\partial P}{\partial V} + \frac{1}{\tau} \left( 1 - \frac{\partial V^e}{\partial V} \right) \right] + i\kappa \rho_e \left[ -\frac{i\kappa}{\rho_e} \frac{\partial P}{\partial \rho} + \frac{1}{\tau} \frac{\partial V^e}{\partial \rho} \right] = 0
\] (15.196)
and has the two solutions
\[
\tilde{\lambda}_\pm(\kappa) = -\frac{1}{2\tau(\kappa)} \pm \sqrt{\Re(\kappa) \pm i|\Im(\kappa)|}
\] (15.197)

with
\[
\frac{1}{\tau(\kappa)} = \frac{1}{\tau} \left( 1 - \frac{\partial V^e}{\partial V} \right) \geq 0, \quad \text{WHY } V^e \partial V^e/\partial V < 1\
\Re(\kappa) = \frac{1}{4\tau^2} - \frac{\kappa^2}{\rho_e^2} \left( \frac{\partial P}{\partial V} \right)^2,
\pm|\Im(\kappa)| = \frac{\kappa \rho_e}{\tau} \left| \frac{\partial V^e}{\partial \rho} \right| - \frac{\kappa}{4\rho_e \tau} \left| \frac{\partial P}{\partial V} \right|,
\] (15.198)
where we have assumed the typical case \( \partial V^e(\rho)/\partial \rho \leq 0, \partial P/\partial V < 0 \) (see Exercise ?? of Chapter ??).

A transition from stable to unstable behavior occurs under the condition
\[
\lambda_+(\kappa) = -\frac{1}{2\tau} + \sqrt{\frac{1}{2} \left( \sqrt{\Re^2 + \Im^2} + \Re \right)} = 0,
\] (15.199)
which again implies
\[
\frac{1}{16\tau^4} = \frac{\Re}{4\tau^2} + \frac{\Im^2}{4},
\] (15.200)
see Eq. (5.31). Inserting the above abbreviations, we eventually find
\[
\frac{\kappa^2}{4\tau^2} \left( \frac{\partial P}{\partial \rho} + \frac{1}{\rho_e^2} \left( \frac{\partial P}{\partial V} \right)^2 \right) = \frac{1}{4} \left( \frac{\kappa \rho_e}{\tau} \left| \frac{\partial V^e}{\partial \rho} \right| + \frac{\kappa}{4\rho_e \tau} \left| \frac{\partial P}{\partial V} \right| \right)^2.
\] (15.201)

From this, we can derive the instability condition
\[
\rho_e \left| \frac{\partial V^e}{\partial \rho} \right| > \frac{1}{4\rho_e} \left( \frac{\partial P}{\partial V} \right) + \frac{1}{\rho_e} \left( \frac{\partial P}{\partial V} \right)^2 \left( 1 + \left| \frac{\partial V^e}{\partial V} \right| \right),
\] (15.202) 

where we have assumed \( \partial V^e/\partial V < 0 \), which can be motivated by the gas-kinetic formula
\[ V^e(\rho, V) = V^0 - \tau(1 - p(\rho))|\chi(\rho)| \rho \theta(\rho, V) \] and Eq. (2.84) [?, ?]. Herein, \( p(\rho) > 0 \) reflects the density-dependent overtaking probability, while \( \chi(\rho) \) is a density-dependent factor reflecting the scarcity of space due to the finite space requirements of cars. As a consequence of the inequality (15.202), we can state that the speed-dependence of the traffic pressure \( P \) and the dynamic equilibrium velocity \( V^e \) tends to make the traffic flow more stable with respect to perturbations. The speed-dependence also resolves problems related to the fact that \( \partial P/\partial \rho \) may become...
negative in a certain density range. This would imply a negative discriminant of the square root, if the negative contribution $\partial P/\partial \rho < 0$ was not compensated for by $(\partial P/\partial V)^2/(\rho_e^2)$.

In case of Payne’s macroscopic traffic model, i.e. for a solely density-dependent velocity-density relation $V_e(\rho)$ and the density-dependent pressure gradient $dP/\partial \rho = |dV_e/\partial \rho|/(2\tau) \geq 0$ (see Eq. (2.62)), the above inequality simplifies considerably. Then, we just find the instability condition

$$\rho_e \left| \frac{dV_e(\rho_e)}{d\rho} \right| > \frac{1}{2\rho_e \tau}. \tag{15.203}$$

Again, traffic flow becomes unstable if the equilibrium velocity $V_e(\rho)$ decreases too rapidly with an increase in the density. LOESUNG FUER ALLGEMEINE PRESSURE RELATION???

**Exercise 5.4**

In order to compare the instability conditions (5.57) for the optimal velocity car-following model with the instability condition (15.203) for Payne’s macroscopic traffic model, let us consider

$$\rho = 1/d, \quad V_e(\rho) = v_e(1/\rho), \tag{15.204}$$

and

$$\left| \frac{dV_e(\rho)}{d\rho} \right| = -dV_e(\rho) = -\frac{dv_e(1/\rho)}{dd} = -\frac{dv_e(d)}{dd} \cdot \frac{1}{\rho^2}. \tag{15.205}$$

Inserting this and $\rho_e = 1/d_e$ into Eq. (15.203) gives

$$\rho_e \left| \frac{dV_e(\rho_e)}{d\rho} \right| = \frac{1}{\rho_e} \frac{dv_e(d_e)}{dd} > \frac{1}{2\rho_e \tau} \quad \text{or} \quad \frac{dv_e(d_e)}{dd} > \frac{1}{2\tau}. \tag{15.206}$$

which agrees with the inequality (5.57). Therefore, the instability conditions of the optimal velocity and the Payne model are identical. The reason for this is the fact that the Payne model is the macroscopic equivalent of the optimal velocity car-following model, see Exercise ???.

**Exercise 5.5**

Let us start with the representation (5.18) of the general solution of the linearized system of equations, focussing (for simplicity) on the case $\lambda(\omega) = 0$ and assuming a “Gaussian wave packet” with

$$\delta \rho_0(\kappa) = \frac{e^{-(\kappa - \kappa_0)^2/(2\theta)}}{\sqrt{2\pi\theta}}. \tag{15.207}$$

Via the linear Taylor approximation $\omega_1(\kappa) = \omega_1(\kappa_0) + c_l \Delta \kappa$ with $c_l = d\omega_1(\kappa_0)/d\kappa$ and $\Delta \kappa = (\kappa - \kappa_0)$, we get
\[ \delta \rho(x, t) = \sum_{l \in \{+,-\} \cdots -\infty} ^{\infty} \int \frac{e^{-\left((\kappa-\kappa_0)^2/(2\theta)\right)}}{\sqrt{2\pi\theta}} e^{i(\kappa x - \omega_l (\kappa) t)} \]

\[ = \sum_{l \in \{+,-\}} e^{i(\kappa_0 x - \omega_l (\kappa_0) t)} \int _{-\infty} ^{\infty} d\Delta \kappa \frac{e^{-\left((\Delta \kappa)^2/(2\theta)\right)}}{\sqrt{2\pi\theta}} e^{i\Delta \kappa x - \omega (\kappa - c_l) t} \]

\[ = \sum_{l \in \{+,-\}} e^{i(\kappa_0 x - \omega_l (\kappa_0) t)} \int _{-\infty} ^{\infty} d\Delta \kappa \frac{e^{-\left(\Delta \kappa - \theta (x - c_l t))^2/(2\theta)\right)}}{\sqrt{2\pi\theta}} e^{-\theta (x - c_l t)^2/2} \]

\[ = \sum_{l \in \{+,-\}} e^{i(\kappa_0 x - \omega_l (\kappa_0) t)} e^{-\theta (x - c_l t)^2/2} . \tag{15.208} \]

While the single waves of frequency \( \omega_l (\kappa) \) move at speed \( x/t = \omega_l (\kappa)/\kappa \), it turns out that their superposition behaves like a wave with frequency \( \omega_l (\kappa_0) \) and speed \( x/t = \omega_l (\kappa_0)/\kappa_0 \). However, the waves packet or, more exactly speaking, its amplitude \( e^{-\theta (x - c_l t)^2/2} \) is moving with the group velocity \( x/t = c_l = d\omega_l (\kappa)/d\kappa \). Note that the case \( c_l > \omega_l (\kappa_0)/\kappa_0 \), in which the group velocity is greater than the wave velocity, is possible! It is called “anomalous dispersion”.

While the variance in the wave numbers according to Eq. (15.207) is \( \theta \), the variance of the amplitude \( e^{-\theta (x - c_l t)^2/2} \) in Eq. (15.208) is \( 1/\theta \), i.e. the product of variances is 1. This establishes something like an “uncertainty principle” between the localization in wave number space and in real space, i.e. a sharp localization (small variance) in wave number space implies a wide distribution in real space and vice versa.

### Exercises of Chapter 6

#### Exercise 6.1

Calculating the first derivatives of the mathematical functions and considering \( d\sin(x)/dx = \cos(x), d\cos(x)/dx = -\sin(x), d\exp(x)/dx = \exp(x) \), \( \sqrt{b+x} = (b+x)^{1/n} \), and \( dx^m/dx = m x^{m-1} \), we find

\[ \frac{d\sin(x)}{dx} = \cos(x), \quad \frac{d^2 \sin(x)}{dx^2} = -\sin(x), \quad \frac{d^3 \sin(x)}{dx^3} = -\cos(x) \]

\[ \frac{d\exp(x)}{dx} = \exp(x), \quad \frac{d^2 \exp(x)}{dx^2} = \exp(x), \quad \frac{d^3 \exp(x)}{dx^3} = \exp(x) \]

\[ \frac{d(b+x)^{\frac{1}{n}}}{dx} = \frac{1}{n} (b+x)^{-\frac{1}{n}}, \quad \frac{d^2 (b+x)^{\frac{1}{n}}}{dx^2} = \frac{1}{n} (b+x)^{-\frac{2}{n}}, \quad \frac{d^3 (b+x)^{\frac{1}{n}}}{dx^3} = \frac{1}{n} (b+x)^{-\frac{3}{n}} \tag{15.209} \]

Taking into account \( \sin(0) = 0, \cos(0) = 1, \exp(0) = 1 \), and \( (b+0)^{1/n} = b^{1/n} = \sqrt{b} \), we get the first terms of the respective Taylor series (6.1) expansion as
\[
\sin(x) = x - \frac{x^3}{3!} + \ldots ,
\]
(15.210)
\[
\cos(x) = 1 - \frac{x^2}{2!} + \ldots ,
\]
(15.211)
\[
\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots ,
\]
(15.212)
\[
\sqrt[n]{b \pm x} = \sqrt[n]{b} \left(1 \pm \frac{x}{nb} \pm \frac{(n-1)x^2}{2!(nb)^2} \pm \frac{(2n-1)(n-1)x^3}{3!(nb)^3} - \ldots \right) .
\]
(15.213)

One can actually show that
\[
\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} ,
\]
(15.214)
\[
\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} ,
\]
(15.215)
\[
\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} .
\]
(15.216)

Due to \((-x)^{2n+1} = (-1)^{2n+1} x^{2n+1} = -x^{2n+1}\) and \((-x)^{2n} = (-1)^{2n} x^{2n} = x^{2n}\), we have \(\cos(-x) = \cos(x)\), but \(\sin(-x) = -\sin(x)\). Furthermore, with \(i^{2n} = (-1)^n\) and \(i^{2n+1} = i(-1)^n\), we obtain the Eulers formula
\[
\exp(i\varphi) = \sum_{n=0}^{\infty} \frac{i^n \varphi^n}{n!} = \sum_{l=0}^{\infty} \frac{i^{2l} \varphi^{2l}}{(2l)!} + \sum_{l=0}^{\infty} \frac{i^{2l+1} \varphi^{2l+1}}{(2l+1)!} \\
= \sum_{l=0}^{\infty} \frac{(-1)^l \varphi^{2l}}{(2l)!} + i \sum_{l=0}^{\infty} \frac{(-1)^l \varphi^{2l+1}}{(2l+1)!} \\
= \cos(\varphi) + i \sin(\varphi) .
\]
(15.217)

Finally, the series expansion
\[
\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n,
\]
(15.218)

for \(|x| < 1\) can be easily shown via
\[
(1-x) \sum_{n=0}^{m} x^n = \sum_{n=0}^{m} x^n - \sum_{n=1}^{m+1} x^n = x^m - x^{m+1} = 1 - x^{m+1} ,
\]
(15.219)

which implies
\[
\sum_{n=0}^{m} x^n = \frac{1 - x^{m+1}}{1-x}
\]
(15.220)

and, as \(\lim_{m \to \infty} x^{m+1} = 0\) for \(|x| < 1\), the geometric series (15.218).
Exercise 6.2

From \(\tan(\varphi) = \sin(\varphi)/\cos(\varphi)\) follows, by taking squares and applying \(\cos^2(\varphi) = 1 - \sin^2(\varphi)\):

\[
\tan^2(\varphi) = \sin^2(\varphi) \left[1 + \tan^2(\varphi)\right] = \left[1 - \cos^2(\varphi)\right] \left[1 + \tan^2(\varphi)\right].
\]  
(15.221)

Consequently, we have

\[
\sin(\varphi) = \pm \frac{\tan(\varphi)}{\sqrt{1 + \tan^2(\varphi)}}
\]  
(15.222)

and

\[
\cos^2(\varphi) = 1 - \frac{\tan^2(\varphi)}{1 + \tan^2(\varphi)} = \frac{1}{1 + \tan^2(\varphi)}.
\]  
(15.223)

Exercise 6.3

We derive the summation theorems of trigonometric functions using the two different presentations of complex numbers given by Eq. (5.12) and the addition theorem for the exponential function:

\[
\cos(\alpha \pm \beta) + i \sin(\alpha \pm \beta) = e^{i(\alpha \pm \beta)} = e^{i\alpha} e^{\pm i\beta} = \\
\left[\cos(\alpha) + i \sin(\alpha)\right] \left[\cos(\beta) \pm i \sin(\beta)\right] = \\
= \cos(\alpha) \cos(\beta) \mp \sin(\alpha) \sin(\beta) + \\
i \left[\sin(\alpha) \cos(\beta) \pm \cos(\alpha) \sin(\beta)\right].
\]  
(15.224)

The real and imaginary parts imply the well-known theorems

\[
\cos(\alpha \pm \beta) = \cos(\alpha) \cos(\beta) \mp \sin(\alpha) \sin(\beta),
\]  
(15.225)

\[
\sin(\alpha \pm \beta) = \sin(\alpha) \cos(\beta) \pm \cos(\alpha) \sin(\beta).
\]  
(15.226)

For \(\beta = \alpha\), the first one of these particularly implies

\[
1 = \cos(0) = \cos^2(\alpha) + \sin^2(\alpha).
\]  
(15.227)

Furthermore, one can show

\[
\sin(\alpha) \sin(\beta) = \frac{1}{2} \left[\cos(\alpha - \beta) - \cos(\alpha + \beta)\right],
\]  
(15.228)

\[
\sin(\alpha) \cos(\beta) = \frac{1}{2} \left[\sin(\alpha + \beta) + \sin(\alpha - \beta)\right],
\]  
(15.229)

\[
\cos(\alpha) \sin(\beta) = \frac{1}{2} \left[\sin(\alpha + \beta) - \sin(\alpha - \beta)\right],
\]  
(15.230)

\[
\cos(\alpha) \cos(\beta) = \frac{1}{2} \left[\cos(\alpha + \beta) + \cos(\alpha - \beta)\right].
\]  
(15.231)
Exercise 6.4

Assume two infinite series \( \sum_{m=0}^{\infty} a_m \) and \( \sum_{n=0}^{\infty} b_n \). Given the required convergence properties, the product of these infinite series is

\[
\left( \sum_{m=0}^{\infty} a_m \right) \cdot \left( \sum_{n=0}^{\infty} b_n \right) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} a_m b_n.
\]  

(15.232)

We may now rearrange the order of terms, considering that each combination of \( m \) and \( n \) occurs exactly once and that there are exactly \( k + 1 \) products \( a_m b_n \) with \( m + n = k \), namely \( a_0 b_k \), \( a_1 b_{k-1} \), \ldots , \( a_{k-1} b_1 \), \( a_k b_0 \). Therefore, we find

\[
\left( \sum_{m=0}^{\infty} a_m \right) \cdot \left( \sum_{n=0}^{\infty} b_n \right) = \sum_{k=0}^{\infty} \sum_{l=0}^{k} a_l b_{k-l},
\]  

(15.233)

where

\[
e_k = \sum_{l=0}^{k} a_l b_{k-l}
\]  

(15.234)

is called the Cauchy product. Relationship (15.233) is also known as discrete convolution theorem. If the infinite sums start from \( 1 \) rather than \( 0 \), with \( a_0 = 0 \) and \( b_0 = 0 \) we get the following relationship from Eq. (15.233):

\[
\left( \sum_{m=1}^{\infty} a_m \right) \cdot \left( \sum_{n=1}^{\infty} b_n \right) = \sum_{k=1}^{\infty} \sum_{l=1}^{k-1} a_l b_{k-l}.
\]  

(15.235)

Note that we do not get any contribution for \( k = 1 \), as the inner sum does not contain any terms.

Above, we have sorted the terms according to identical values of \( k = m + n \). If we have to arrange the terms of the series’ product according to \( k = m - n \) instead, we get

\[
\left( \sum_{m=1}^{\infty} a_m \right) \cdot \left( \sum_{n=1}^{\infty} b_n \right) = \sum_{k=1}^{\infty} \sum_{l=1}^{k} a_l b_{l-k},
\]  

(15.236)

where we have first listed contributions with \( l = k - k = 0 \), then contributions with \( (l + k) - l = k > 0 \), and finally contributions with \( l - (l + k) = -k < 0 \).

Exercise 6.5

Let us assume a circular road with two cars. In this situation, as the distances add up to the length \( L \) of the road \( (d_1(t) + d_2(t) = L) \), we have \( x_{j-1}(t) = \delta d_{j-1}(t) = -\delta d_j(t) = -x_j(t) \). Consequently, we have \( x_j(t) - x_{j-1}(t) = 2x_j(t) \) and \( \left[ x_j(t) \right]^2 - \left[ x_{j-1}(t) \right]^2 = 0 \). Consequently, Eq. (6.6) becomes
\[
\frac{d^2 x_j(t)}{dt^2} + 2\gamma \frac{dx_j(t)}{dt} + 2(\omega_0)^2 x_j(t) = 0,
\]
which implies the characteristic polynomial
\[
\tilde{\lambda}^2 + 2\gamma \tilde{\lambda} + 2(\omega_0)^2 = 0.
\]
The real part of the solution
\[
\tilde{\lambda}_\pm = -\gamma \pm \sqrt{\gamma^2 - 2(\omega_0)^2}
\]
is always negative, i.e. the stationary solution is stable with respect to perturbations.

Exercises of Chapter 7

Exercise 7.1

In the case \(\rho_{c2} < \rho_{\text{max}}\), the maximum flow \(Q_{\text{max}}\) would be linearly unstable, so that it could be reached only for short time periods. The maximum flow that could be maintained over longer time periods would rather be given by \(Q_{c2}\).